

The causal boundary of wave-type spacetimes

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ABSTRACT: A complete and systematic approach to compute the causal boundary of wave-type spacetimes is carried out. The case of a 1-dimensional boundary is specially analyzed and its critical appearance in pp-wave type spacetimes is emphasized. In particular, the corresponding results obtained in the framework of the AdS/CFT correspondence for holography on the boundary, are reinterpreted and very widely generalized.

Technically, a recent new definition of causal boundary is used and stressed. Moreover, a set of mathematical tools is introduced (analytical functional approach, Sturm-Liouville theory, Fermat-type arrival time, Busemann-type functions).

KEYWORDS: Penrose limit and pp-wave background, Classical Theories of Gravity, Differential and Algebraic Geometry.

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Contents

1. Introduction	2
2. Wave-type spacetimes	5
3. The causal boundary of spacetimes	6
3.1 Classical approach	6
3.2 A recent new approach	8
3.3 TIP's as past of lightlike curves	9
4. Fermat's arrival function and functional approach	11
5. Conditions on function F and functional \mathcal{J}	13
6. Non-distinguishing Mp-waves	19
6.1 The general result	20
6.2 Some remarkable examples	20
7. Boundaries in strongly causal Mp-waves	22
7.1 General expressions for $P, F, \uparrow P, \downarrow F$	22
7.2 Boundary for M complete, $ F $ at most quadratic	26
8. Mp-waves with natural 1-dim. $\partial\mathcal{M}$	28
8.1 Collapsing to i^\pm	28
8.2 Case asymptotically quadratic	29
8.3 Plane waves	31
9. Higher dimensionality of $\partial\mathcal{M}$	33
9.1 Criticality of $\lambda = 1/2$ for 1-dimensionality	34
9.2 Static and Minkowski type Mp-waves	35
9.3 The case $-F$ quadratic	36
10. Conclusions	37
A. Technical bounds for functionals	37

1. Introduction

There are several motivations for the recent interest on the boundary of wave type spacetimes. Firstly, there are important reasons for string theory, because of the AdS/CFT correspondence of plane waves and the holographic role of its boundary. But there are also reasons from the viewpoint of General Relativity, apart from the obvious interest in the properties of a classical spacetime. In fact, the old problem on the consistency of causal boundaries and its relation with conformal boundaries is put forward by pp-waves and stimulates its full solution. Very roughly, the main results can be summarized as follows (see also references therein):

- Plane waves yield exact backgrounds for string theory as all their scalar curvature invariants vanish. Thus, they correspond to exact conformal theories, and in some cases can be explicitly quantized [1, 29, 39].
- Taking into account the well-known result that any spacetime has a plane wave as a limit along any lightlike geodesic (Penrose, [44]), Berenstein, Maldacena and Nastase [5] related string theory on maximally supersymmetric 10 dimensional plane waves to 4 dimensional field theory.
- More precisely, Penrose limit on a lightlike geodesic on $\text{AdS}_5 \times S^5$, which rotates on the S^5 was considered. Blau, Figueroa-O’Farrill, Hull and Papadopoulos [9] constructed the limit plane wave and identified its dual in the field theory. Berenstein and Nastase [6] studied the asymptotic *conformal* boundary of this plane wave, finding that it is 1-dimensional. This fact not only was not regarded as pathological, but it suggested that such a plane wave possesses a holographic dual description in terms of quantum mechanics on its boundary –a similar picture to CFT dual to an asymptotically AdS space.
- Marolf and Ross [36] studied the *causal* boundary of that plane wave. There are interesting reasons to use this more sophisticated boundary. On one hand, it is intrinsic to the spacetime and systematically determined. On the other, this approach is applicable to any plane wave or spacetime, not only to conformally flat ones. Essentially, these authors reobtained the 1-dimensional character for the causal boundary of Berenstein and Nastase’s, and, *surprisingly*, obtained other relevant cases of plane waves with this same behavior (as it was independent on the number of positive eigenvalues for the quadratic form F , assuming the existence of at least one). What is more, their results suggested a redefinition of classical causal boundary [37], as this old concept was known to have some undesirable properties.
- In [19] the authors studied systematically the causal structure of wave-type spacetimes (the general family (2.1) below). We showed that this structure depends dramatically on the value of the characteristic coefficient F of the metric. In particular, when F is “at most quadratic” (as in classical plane waves) the spacetime becomes strongly causal, but when it is “superquadratic” the wave is non-distinguishing and

the causal boundary makes no sense. Hubeny, Rangamani and Ross [33] pointed out that this is the case of the pp-wave which gives rise to the $\mathcal{N} = 2$ sine-Gordon string world-sheet; moreover, they also studied other properties on causality (as the existence of time functions) and boundaries for some specific pp-wave backgrounds [30, 32–34].

There are also two technical questions which are worth of pointing out here. First, the systematic study of the causal boundary in [18], starting at the cited original idea [37], which seems to yield a definitive answer to the problem of the identifications between future and past ideal points, as well as an appropriate topology on the boundary. Second, the solution of the so-called “folk problems of smoothability” which yield consistency to the full causal ladder of causality, including the equivalence between stable causality and the existence of a time function [7, 42, 8].

The aim of the present article is to study systematically the causal boundary of wave-type spacetimes. Recall that, essentially, Marolf and Ross [36, 37] studied locally symmetric plane waves ($F(x, u) \equiv F(x)$, F quadratic form), and Hubeny and Rangamani [30] studied particular cases of plane waves, as well as some pp-waves, extracting some heuristic conclusions. But more precise and general results about the structure of the boundaries are missing there.

Summing up, our motivation is threefold: first to conclude the study in [36, 30], originated by applications on strings, second to conclude the study of causality of pp-wave type spacetimes initiated in [19, 13], and third to check and support the new concept of causal boundary in [37, 18]. Our approach can be summarized as follows.

In section 2 we introduce the general class of wave-type spacetimes, namely Mp-waves $\mathcal{M} = M \times \mathbb{R}^2$, to be considered. Other properties of these spacetimes (geodesics, completeness, causal hierarchy) were studied in [13, 19]; some changes of notation are made here.

In section 3 the framework of causal boundaries is introduced. First, the original Geroch, Kronheimer and Penrose (GKP) boundary of TIP’s and TIF’s [24] is recalled section 3.1. The recent progress on this boundary [37, 18] applicable here is summarized in section 3.2. This includes the characterization of ideal points as certain pairs (P, F) of TIP’s and TIF’s (which involves their common futures and pasts $\uparrow P, \downarrow F$), the induced causal relation and the topology of the boundary. Finally, a simple, but general, technical property of TIP’s and TIF’s is proved in section 3.3. Essentially, this property means that TIP’s and TIF’s can be regarded as pasts or futures of certain (non necessarily geodesic) inextendible lightlike curves (proposition 2); its version for Mp-waves (corollary 1) will simplify the functional approach to be used later.

In section 4 we introduce an arrival time function with analogies to classical Fermat’s one [45]. This function allows to introduce a functional $\mathcal{J}_{u_0}^{\Delta u}$ in the space of curves on the spatial M part (essentially, in the set of curves $x(u)$ which connect each two prescribed points $x_0, x_1 \in M$ parametrized by the “ u -quasitime” $u \in [u_0, u_0 + \Delta u]$, where $(x, u, v) \in M \times \mathbb{R}^2$). The infimum of $\mathcal{J}_{u_0}^{\Delta u}$ characterizes which points can be causally joined with each $(x_0, u_0, v_0) \in \mathcal{M}$. This approach, on one hand, allows to introduce techniques and results from functional analysis (some required ones will be developed in the appendix). On the

other, it clarifies the causal structure of Mp-waves; for example, the inexistence of horizons (claimed in [31] and strongly supported in [20]) becomes now apparent (remark 2).

In section 5 we introduce two technical conditions **(H1)**, **(H2)** on the Mp-wave in terms of functional \mathcal{J} (definition 2), and relate them to the qualitative behavior of the characteristic metric coefficient F . Very roughly, the idea is as follows. Each M -curve x determines univocally a lightlike curve type $(x(u), u, v(u)), u \in [u_0, u_0 + \Delta u]$. Assume that the lightcones become opened fast along the lightlike curves generated in one M -direction (or even just along a sequence $\{x_m\}_m$ of M -loops). Due to the structure of the Mp-wave, if this happens for arbitrarily small values of Δu (as formally expresses **(H2)**) then the future of all the points (x, u, v) with the same $u = u_0$ collapses. So, the Mp-wave will be non-distinguishing, and no causal boundary can be defined. Now, assume that the Mp-wave is causally well-behaved and, so, this property does not hold for arbitrarily small Δu . If the property still holds for values of Δu greater than some constant $\Delta_0 > 0$ (as expresses **(H1)**), then the collapse will happen at the level of the TIP's, i.e.: lightlike curves with unbounded coordinate u will generate the same ideal points i^+, i^- .

As conditions **(H1)**, **(H2)** are formulated directly on the functional, they become very technical. Nevertheless, we also define the typical behaviors of F at infinity: super, at most, and sub quadratic (these are general bounds on the growth of $F(\cdot, u)$, depending arbitrarily on u) as well as λ -asymptotically quadratic (such a bound is also restrictive on u). We showed in [19] how some of these behaviors determine the position in the causal ladder of the Mp-wave. Now, we show (lemmas 4, 5) how some of them (superquadratic, λ -asymptotically quadratic with $\lambda > 1/2$) yield naturally conditions **(H2)**, **(H1)**, which will determine its boundary. The results are very accurate, as shown by the bound $\lambda > 1/2$, which comes from Sturm-Liouville theory (see remark 5 and section 9.1). Nevertheless, we emphasize that the technical behaviors **(H1)**, **(H2)** involve only some M -direction. Thus, one can easily yield results more general than stated. In fact, in lemma 5(ii) condition **(H1)** is proved for (non-necessarily locally symmetric) plane waves such that one of the eigenvalues of F is positive; this lies in the core of the surprising result by Marolf and Ross [36] cited above.

In section 6 we prove how **(H2)** forbids the Mp-wave to be distinguishing (section 6.1). This may be somewhat unexpected, and some examples in [30] are revisited (section 6.2).

In section 7 the explicit construction of the ideal points for any strongly causal Mp-wave is carried out. This is done in full generality in section 7.1, where the main result (theorem 2) is expressed in terms of two "Busemann type functions" b^\pm previously introduced (propositions 4, 5). Notice that Busemann functions appear naturally when TIP's or TIF's are computed in simple (standard static) spacetimes, [26]. Now, the more elaborated function b^- plays a similar role to such a Busemann function, and the new function b^+ is introduced to deal with the sets $\uparrow P, \downarrow F$ required for the total causal boundary [37, 18]. Moreover (section 7.2), when $|F|$ is at most quadratic (and, thus, \mathcal{M} is necessarily strongly causal) and M complete, a special simplification of the terminal sets $P, F, \uparrow P, \downarrow F$ occurs. (We emphasize the necessity of the at most quadratic behavior for $|F|$, which was dropped in previous literature, remark 8.) In fact, a natural lightlike ideal line in each boundary $\hat{\partial}\mathcal{M}, \check{\partial}\mathcal{M}$ (parametrized by $u_\infty, |u_\infty| < \infty$ in theorem 3, remark 9) appears. Nevertheless,

the boundary $\partial\mathcal{M}$ may be higher dimensional, because the lightlike curves with unbounded coordinate u ($|u_\infty| = \infty$) may still generate infinitely many ideal points.

However, in section 8 we show that, when additionally **(H1)** holds, then all (future) lightlike curves with $u \nearrow \infty$ generate the same ideal point i^+ , so a 1-dimensional boundary is expected (section 8.1). In particular, *when F is λ -asymptotically quadratic with $\lambda > 1/2$ the boundary is two copies of a 1-dimensional lightlike line, with some eventual identifications* (section 8.2). Moreover the special case of plane waves is compared carefully with previous results and techniques (remark 11 and below).

In section 9 we consider subquadratic F 's and emphasize the critical character of the 1-dimensional boundary. Recall that, essentially, such a boundary corresponds to a ($\lambda > 1/2$)-asymptotically quadratic behaviour of F , and the boundary makes no sense under a faster (superquadratic) growth. In section 9.1 we construct an explicit example with higher dimensional boundary in the limit case $\lambda = 1/2$. So, the 1-dimensional boundary can no longer be expected.

Higher dimensionality is expected specially in the (globally hyperbolic) subquadratic case section 9.2. Notice that the case $M = \mathbb{R}^n, F \equiv 0$ corresponds to Lorentz-Minkowski \mathbb{L}^{n+2} (for arbitrary M , corresponds to a standard static spacetime). If $|F(\cdot, u)|$ is upper bounded for each u , then the spacetime becomes “isocausal” (in the sense of García-Parrado and Senovilla [22]) to \mathbb{L}^{n+2} and, thus, the causal boundary is expected to be $(n + 1)$ -dimensional.

Finally, in section 9.3 we discuss and extend Marolf and Ross’ result [36, section 3.1] for plane waves with negative eigenvalues. Concretely, we reobtain that the Mp-wave is conformal to a region of \mathbb{L}^{n+2} bounded by two lightlike hyperplanes, even when F depends on u . Nevertheless, a discussion shows that the causal and conformal boundaries differ in this case: the former has two connected pieces (a future boundary and a past one); the latter, which is necessarily compact, is connected and includes implicitly properties at spacelike infinity (compare with [38]).

We finish emphasizing some conclusions in section 10, including a table of results, and providing some technical bounds on some functionals in the appendix. Along the paper, four figures have been also included as a guide for the reader.

2. Wave-type spacetimes

The authors, in collaboration with A.M. Candela, introduced and studied systematically [13, 19, 20] the following class of spacetimes, which widely generalize classical pp-waves (and, thus, plane waves):

$$\begin{aligned} (\mathcal{M}, \langle \cdot, \cdot \rangle_L) \quad \mathcal{M} = M \times \mathbb{R}^2 \\ \langle \cdot, \cdot \rangle_L = \langle \cdot, \cdot \rangle - F(x, u) du^2 - 2 du dv. \end{aligned} \tag{2.1}$$

Here $(M, \langle \cdot, \cdot \rangle)$ is any smooth Riemannian (C^∞ , positive-definite, connected) n -manifold, the variables (u, v) are the natural coordinates of \mathbb{R}^2 and $F : M \times \mathbb{R} \rightarrow \mathbb{R}$ is any smooth scalar field. M will not be assumed to be complete a priori, and will be said *unbounded* if it is non-compact with points at arbitrary long distances (i.e., it has infinite diameter).

These spacetimes were named just PFW (“plane fronted waves”) in some previous references but, according to the more careful notation in the survey [23], they will be considered as (a type of) *Mp-waves*. We also introduce some changes of conventions and notations in order to make a better comparison with references such as [36, 30, 47]. In particular, function F here replaces $-H$ in previous references. We will choose once for ever a point $\bar{x} \in M$. Then, if d is the natural distance associated to the Riemannian metric $\langle \cdot, \cdot \rangle$, we put

$$|x| = d(x, \bar{x}) \quad \forall x \in M. \tag{2.2}$$

Elementary properties of these spacetimes are the following. Vector field ∂_v is parallel and lightlike, and the time-orientation will be chosen to make it future-directed. Thus, for any future-directed causal curve $\gamma(s) = (x(s), u(s), v(s))$, $s \in I$ (I interval)

$$\dot{u}(s) = -\langle \dot{\gamma}(s), \partial_v \rangle_L \geq 0, \tag{2.3}$$

being the inequality strict if $\gamma(s)$ is timelike (and analogously for a past-directed curve). Using this inequality and the fact that $\nabla u = -\partial_v$, it follows that any such Mp-wave is causal. The slices $u \equiv \text{constant}$ are degenerate, with radical $\text{Span } \partial_v$. Then, all the hypersurfaces (non-degenerate n -submanifolds of \mathcal{M}) of one such a slice which are transverse to ∂_v , become isometric to open subsets of M . The fronts of the wave (2.1) will be defined as the (whole) slices at constant u, v .

3. The causal boundary of spacetimes

We refer to well-known references such as [43, 2, 27, 51] and specially the recent review [42] for notation and background on causality. For the specific approach on causal boundaries, we refer to [18] and references therein.

3.1 Classical approach

Let $\mathcal{M} \equiv (\mathcal{M}, g)$ be a spacetime, endowed with a time-orientation (implicitly assumed) and, thus, the causal \leq (strict causal $<$) and chronological \ll relations. As usual, causal elements in any open subset $U \subseteq \mathcal{M}$, regarded as a spacetime in its own right, will be denoted such as $<_U$, $J^+(p, U)$, etc. A continuous curve $\gamma : [0, b) \rightarrow \mathcal{M}$ is called *future-directed causal* if, for each $s \in [0, b)$, there exists a convex neighborhood (i.e. a (starshaped) normal neighborhood of all its points) U of $\gamma(s)$ such that, whenever $s' \in (s, b)$ (resp. $s' \in [0, s)$) satisfies that $\gamma([s, s'])$ (resp. $\gamma([s', s])$) is included in U , then $\gamma(s) <_U \gamma(s')$ (resp. $\gamma(s') <_U \gamma(s)$). It is well-known that, up to a reparametrization, such curves are locally Lipschitzian as well as other properties [14, appendix], [42, section 3.5]. This definition (and related properties) are naturally extended not only to the past case, but also to other domains for γ different to $[0, b)$; definitions are also extended to *timelike* curves, with no further mention. A (future or past-directed) causal curve $\gamma : [0, b) \rightarrow \mathcal{M}$ is *piecewise smooth* if there exists a sequence $\{s_i\} \nearrow b$, $s_0 = 0$ such that γ is smooth on each interval $[s_i, s_{i+1}]$ for all i . Notice that, at any (possibly non-smooth) *break* $\gamma(s_i)$, $i > 0$, there are two limit derivatives $\dot{\gamma}(s_i^-)$, $\dot{\gamma}(s_i^+)$, which are causal vectors in the same cone. A piecewise smooth geodesic will be called a *broken* geodesic.

Roughly, the main purpose of the causal completion of a spacetime is to make inextendible timelike curves to end at some point.¹ So, ‘ideal points’ are added to the spacetime, in such a way that any timelike curve has some endpoint in the new extended space (at the original manifold or at an ideal point). To this aim, there will not be any difference if the (timelike) curves are required to be smooth, piecewise smooth or continuous. So, in what follows, all the curves will be piecewise smooth, except when otherwise is said explicitly. The natural level in the causal hierarchy of spacetimes required for the completion of (\mathcal{M}, g) is *strong causality*. In fact, to be (pointwise future or past) distinguishing will be a minimum property in order to recover the points of \mathcal{M} from the general construction, but strong causality will be necessary to recover the topology too, as well as for other technical properties.

In order to describe the completion procedure some terminology is required first. A subset $P \subseteq \mathcal{M}$ is called a *past set* if it coincides with its chronological past $I^-[P]$, that is, $P = I^-[P] := \{p \in \mathcal{M} : p \ll q \text{ for some } q \in P\}$. Given a subset $S \subseteq \mathcal{M}$, we define the *common past* of S as $\downarrow S := I^-[\{p \in \mathcal{M} : p \ll q \ \forall q \in S\}]$. Notice that $I^-[P]$ is always open, and we have chosen the definition of $\downarrow S$ in order to make it open too. A non-empty past set that cannot be written as the union of two proper subsets, both of which are also past sets, is called *indecomposable past set*, IP. An IP which does coincide with the past of some point in \mathcal{M} is called *proper indecomposable past set*, PIP and, otherwise, *terminal indecomposable past set*, TIP. Of course, by replacing the word ‘past’ by ‘future’ we obtain the corresponding notions for *future set*, *common future*, IF, PIF and TIF.

To construct the future causal completion, firstly identify every event $p \in \mathcal{M}$ with its PIP, $I^-(p)$. Then, define the *future causal boundary* $\hat{\mathcal{M}}$ of \mathcal{M} as the set of all TIPs in \mathcal{M} . Therefore, *the future causal completion* $\hat{\mathcal{M}}$ becomes the set of all IPs:

$$\mathcal{M} \equiv \text{PIPs}, \quad \hat{\mathcal{M}} \equiv \text{TIPs}, \quad \hat{\mathcal{M}} \equiv \text{IPs}.$$

Analogously, every event $p \in \mathcal{M}$ can be identified with its PIF, $I^+(p)$, then the *past causal boundary* $\check{\mathcal{M}}$ of \mathcal{M} is the set of all TIFs in \mathcal{M} and thus, *the past causal completion* $\check{\mathcal{M}}$ is the set of all IFs:

$$\mathcal{M} \equiv \text{PIFs}, \quad \check{\mathcal{M}} \equiv \text{TIFs}, \quad \check{\mathcal{M}} \equiv \text{IFs}.$$

In order to define the (total) causal completion, the space $\hat{\mathcal{M}} \cup \check{\mathcal{M}}$ appears obviously. However, it becomes evident that, in order to obtain a reasonably consistent definition: (a) PIP’s and PIF’s must be identified in an obvious way ($I^-(p) \sim I^+(p)$ on $\hat{\mathcal{M}} \cup \check{\mathcal{M}}$ for all $p \in \mathcal{M}$), and (b) the resulting space $\mathcal{M}^\#$ does not provide a satisfactory description of the boundary of \mathcal{M} , because this procedure often attaches two ideal points where we would expect only one (consider the boundary for the interior of a $(n - 1)$ -rectangle in Lorentz-Minkowski \mathbb{L}^n : each point at any timelike side determines naturally both, a TIP and a TIF). There have been many attempts to define additional identifications between elements of $\hat{\mathcal{M}} \cup \check{\mathcal{M}}$ in order to overcome this problem [24, 11, 46, 49, 50], but without totally satisfactory results up to now.

¹In this sense, the name of *chronological completion* would be more appropriate (as in [18]). Nevertheless,

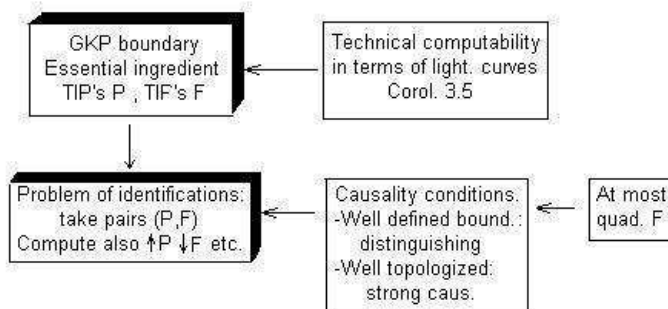


Figure 1: Overall causality framework

3.2 A recent new approach

An alternative procedure to making identifications consists of forming pairs composed by past and future indecomposable sets of \mathcal{M} . This approach, firstly introduced by Marolf and Ross [37], and widely developed in [18], has exhibited satisfactory results for the spacetimes analyzed up to date, and seems specially well-adapted to those ones analyzed in [36, 30]; so, we will adopt this approach in this paper. Even though, as emphasized in [18], there are different choices for the meaning of the total causal boundary once the pairs have been defined, they coincide in most cases and, in particular, in the relevant cases considered here.

Let P (resp. F) be an IP (resp. IF). We say that P is S -related (Szabados related) to F , namely $P \sim_S F$, if P is maximal as IP into $\downarrow F$ and F is maximal as IF into $\uparrow P$. A TIP P can be S -related with more than one TIF F_1, F_2 (take $P = \{(x, t) : |x| < -t\}$ in $\mathcal{M} = \mathbb{L}^2 \setminus \{(0, t) : t \geq 0\}$) or viceversa. Nevertheless, this will not happen in our study (remark 9). Therefore, according to [18, 37], the (total) causal completion $\overline{\mathcal{M}}$ is defined in this case as: the set of pairs (P, F) where P (resp. F) is either a IP (resp. IF) or the empty set and one of the following possibilities happens: (a) $P \sim_S F$, (b) $F \neq P = \emptyset$ and there is no P' such that $P' \sim_S F$, or (c) $P \neq F = \emptyset$ and there is no F' such that $P \sim_S F'$. The (total) causal boundary is the subset $\partial\mathcal{M} \subset \overline{\mathcal{M}}$ containing the pairs (P, F) such that P is not a PIP (and, thus, F is not a PIF [49, proposition 5.1]).

With this definition at hand, it is easy to extend the chronological relation \ll to the completion $\overline{\mathcal{M}}$: (P, F) is *chronologically related* to (P', F') , namely $(P, F) \ll (P', F')$, if $F \cap P' \neq \emptyset$. The properties and absence of problems for this choice are well established [18];

here we will maintain the term *causal completion* to emphasize that some causal elements have been introduced, and in close correspondence with previous literature such as [36, 30].

nevertheless, it is not so easy to give a definitive extension of the causal relation. As the boundary of some waves is sometimes claimed to be null in a rather intuitive way, we will adopt here simple definitions which will formalize this, and postpone to future work other subtleties in more general cases. We say that (P, F) is *causally related* to (P', F') , namely $(P, F) \leq (P', F')$, if $F' \subseteq F$ and $P \subseteq P'$, at least one of them not trivially (i.e., without involving the empty set, $P \neq \emptyset$ or $F' \neq \emptyset$). This is a canonical choice to define a causal relation from a chronological one (taken also in [37]; see [42, definition 2.22, theorem 3.69] for a discussion). If we only impose that one of these two inclusions hold (not trivially), we will say (P, F) is *weakly causally related* to (P', F') , written $(P, F) \leq_w (P', F')$. It is easy to check that the latter definition does not imply the former one (consider in $\mathbb{L}^2 \setminus \{(x, t) \in \mathbb{R}^2 : x \leq 0\}$ the ideal point associated to $(0, 0)$ and the pair associated to the point $(-1, 1)$). Finally, (P, F) and (P', F') are *(weakly) horismotically related* if they are (weakly) causally, but not chronologically, related.

The topology of the spacetime can be also extended to the completion. We will adopt the *chronological topology* introduced in [18]. This topology is defined in terms of the following *limit operator* L : given a sequence $\sigma = \{(P_n, F_n)\} \subset \overline{\mathcal{M}}$ and $(P, F) \in \overline{\mathcal{M}}$, we say that $(P, F) \in L(\sigma)$ if²

$$\begin{aligned} P \in \hat{L}(P_n) &:= \{P' \in \hat{\mathcal{M}} : P' \subseteq \text{LI}(P_n) \text{ and } P' \text{ is maximal IP into } \text{LS}(P_n)\} \\ F \in \check{L}(F_n) &:= \{F' \in \check{\mathcal{M}} : F' \subseteq \text{LI}(F_n) \text{ and } F' \text{ is maximal IF into } \text{LS}(F_n)\} \end{aligned}$$

(recall that either P or F can be empty, but not both). Then, the *closed sets* for the chronological topology are those subsets $C \subseteq \overline{\mathcal{M}}$ such that $L(\sigma) \subseteq C$ for any sequence σ in C .

3.3 TIP's as past of lightlike curves

In order to study the pairs in $\partial\mathcal{M}$, it is well-known that any TIP, P , of a strongly causal spacetime can be regarded as $I^-[\rho]$ for some inextendible future-directed timelike curve ρ (see, for example, [2, proposition 6.14]) and analogously for TIF's. Moreover, in this case, it is $\uparrow \rho = \uparrow I^-[\rho]$. Our aim is to show that lightlike broken geodesics are also enough.

Remark 1. *As a previous technicality, recall that when γ is lightlike, then $\uparrow \gamma = \uparrow I^-[\gamma]$ does not necessarily hold: take $\gamma(s) = (s, s)$, $s < 0$ in $\mathbb{L}^2 \setminus \{(0, t) : t \geq 0\}$.*

Nevertheless, this property is ensured when the easily checkable condition $\gamma \subset I^-[\gamma]$ holds.

Proposition 1. *Let $\gamma : [0, b) \rightarrow \mathcal{M}$ be a future-directed (right) inextendible lightlike curve in the strongly causal spacetime \mathcal{M} . If $\gamma \subset I^-[\gamma]$ then $P = I^-[\gamma]$ is a TIP and $\uparrow \gamma = \uparrow P$.*

Proof. Take a sequence $\{s_i\} \nearrow b$. The assumption on γ implies the existence of a subsequence $\{s_{i_k}\}_k$ such that $\gamma(s_{i_k}) \ll \gamma(s_{i_{k+1}})$ for all k . Thus, joining each pair of points by

²By LI and LS we mean the usual inferior and superior limits of sets: i.e. $\text{LI}(A_n) \equiv \liminf(A_n) := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ and $\text{LS}(A_n) \equiv \limsup(A_n) := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. This definition is naturally extended when the range of n is not the set of natural numbers, but a totally ordered set such as the interval $[0, b)$ (this permits to say when a curve γ tends to a boundary point (P, F) without using sequences).

means of a future-directed timelike curve, a piecewise smooth inextendible timelike curve ρ is obtained. Clearly, $I^-[\rho] = I^-[\gamma]$ (and thus, a TIP), $\uparrow \gamma = \uparrow \rho (= \uparrow I^-[\rho])$, and the result follows. ■

Proposition 2. *Let $\rho : [0, b) \rightarrow \mathcal{M}$ be a future-directed causal curve. Then, for any sequence $\{s_i\} \nearrow b$, $s_0 \geq 0$ there exists a broken future-directed lightlike geodesic (with no conjugate points in each unbroken piece) $\gamma : [0, b) \rightarrow \mathcal{M}$ such that $\gamma(s_i) = \rho(s_i)$ for all i and, thus:*

$$I^-[\rho] = I^-[\gamma], \quad \uparrow \rho = \uparrow \gamma.$$

Even more, if (a) $\dim \mathcal{M} \geq 3$, (b) X is any lightlike geodesic vector field and (c) the restriction of ρ to any open interval is not an integral curve of X (up to reparametrization), then γ can be chosen such that $\dot{\gamma}(s)$ is linearly independent of $X_{\gamma(s)}$ for all $s \in [0, b)$. In particular, this holds if ρ is timelike; moreover, in this case $\gamma \subset I^-[\rho]$.

For the proof, notice first:

Lemma 1. *For each $s \in [0, b)$ (resp. $s \in (0, b)$) there exists some $\epsilon > 0$ such that, if $s' \in (s, s + \epsilon)$ (resp. $s' \in (s - \epsilon, s)$) then $\rho(s)$ and $\rho(s')$ can be joined with a broken lightlike geodesic as in proposition 2 with only one break.*

Proof. (Reasoning just for the case $s' > s$). Let U be a convex neighborhood of $p = \rho(s)$. It is known that there exists a globally hyperbolic neighborhood $\tilde{U} \ni p$, $\tilde{U} \subset U$ which is causally convex in U (i.e., such that any causal curve in U with endpoints in \tilde{U} is entirely contained in \tilde{U}), see [42]. Notice that $E^+(p, \tilde{U}) = \partial J^+(p, \tilde{U})$. Let $\epsilon > 0$ such that $\rho([s, s + \epsilon]) \subset \tilde{U}$. For any $s' \in (s, s + \epsilon]$, any past-directed lightlike geodesic β starting at $p' = \rho(s')$ must cross $E^+(p, \tilde{U})$ at some point q (recall that β cannot remain imprisoned in the compact set $J^+(p, \tilde{U}) \cap J^-(p', \tilde{U})$; notice also that, eventually, $q = p'$ or $q = p$ may hold if ρ is lightlike). Thus, the unique (up to reparametrization) broken lightlike geodesic γ in \tilde{U} which goes from p to q and then to p' is the required one.

Even more, in the case $\dim \mathcal{M} \geq 3$ and X geodesic, there are at most two such broken geodesics γ_1, γ_2 which connect p, p' and are integral curves of X at some point (if they existed, one of them γ_1 would be obtained by taking β as the integral curve of X through p' , and the other one γ_2 , analogously starting with an integral curve from p). Thus, it is enough to construct γ by choosing β in a direction different to the velocities of γ_1 and γ_2 on p' . The remainder for the case ρ timelike is straightforward. ■

Proof of proposition 2. Each interval $[s_i, s_{i+1}]$ can be covered by open subsets type $(s - \epsilon, s + \epsilon)$, $(s_{i+1} - \epsilon, s_{i+1})$, $[s_i, s_i + \epsilon)$, with ϵ satisfying the properties of lemma 1. Now, choose δ small enough to make each $(s - \delta, s + \delta) \cap [s_i, s_{i+1}]$ included in one of these open subsets (i.e., δ is taken smaller than a Lebesgue number of the covering) with $s_{i+1} = s_i + k_i \delta$ for some positive integer k_i . Then, the result follows by joining each $\rho(s_i + k\delta), \rho(s_i + (k+1)\delta)$, (for $k = 0, 1, \dots, k_i - 1$ and all i), as in lemma 1. ■

In the case of Mp-waves, broken lightlike geodesics as in proposition 2 for $X = \partial_v$ will be chosen. Summing up, the following result (and its analog for the future case) will be used systematically.

Corollary 1. *Let \mathcal{M} be a strongly causal Mp-wave and P be a TIP. Then, there exists an inextendible future-directed lightlike curve γ (in fact, a broken geodesic without conjugate points) at no point proportional to ∂_v , such that $P = I^-[\gamma]$ and $\uparrow P = \uparrow \gamma$.*

Conversely, if γ is any inextendible future-directed causal curve with $\gamma \subset I^-[\gamma]$ then $P = I^-[\gamma]$ is a TIP and $\uparrow P = \uparrow \gamma$.

4. Fermat's arrival function and functional approach

Vector field ∂_v allows to define an “arrival function” analogous to classical Fermat's time arrival one, as well as an associated functional. In order to carry out the analogy, consider first the simple case of a product spacetime³ ($S \times \mathbb{R}, g = g_S - dt^2$), where (S, g_S) is a Riemannian manifold and ∂_t points out to the future. (Notice that, if $F \equiv 0$, a Mp-wave can be regarded as one such product spacetime with $S = M \times \mathbb{R}$, after a change of the coordinates u, v .) Let $x_0, x_1 \in S, \Delta > 0$. For any piecewise smooth curve $y : [0, \Delta] \rightarrow S$ with endpoints $y(0) = x_0, y(\Delta) = x_1$ a unique future-directed lightlike curve $\gamma(t) = (y(s(t)), t), t \in [0, T]$ can be constructed, being $s(t)$ and $T = T[y]$ determined by $g(\dot{\gamma}, \dot{\gamma}) \equiv 0, s(0) = 0, s(T) = \Delta$. So, if $\mathcal{C} \equiv \mathcal{C}(x_0, x_1; \Delta)$ denotes the set of all such curves $y = y(s)$, a functional

$$\mathcal{J} : \mathcal{C} \rightarrow \mathbb{R}, \quad y \mapsto T[y]$$

is obtained. Now, consider the (future) *time arrival map*

$$T : S \times S \rightarrow \mathbb{R}, \quad (x_0, x_1) \mapsto T(x_0, x_1) := \text{Inf}_{\mathcal{C}} \mathcal{J}.$$

Easily, one has:

$$(x_0, t_0) \ll (x_1, t_1) \quad \Leftrightarrow \quad T(x_0, x_1) < t_1 - t_0.$$

In fact, $T(x_0, x_1)$ is the (Fermat) minimum arrival time of a future-directed lightlike curve from $(x_0, 0)$ to the line $\{x_1\} \times \mathbb{R}$. Notice that in this simple case function T is always finite and continuous, and essentially the same function is obtained if past-directed causal curves are taken (see [47]). Next, our aim is to make a similar construction for any Mp-wave (2.1) but now playing ∂_v the role of ∂_t . The construction can be also generalized to Eisenhart metrics [41]. Previously, observe that formulas (2.3) and (2.1) yield, respectively, the following two lemmas.

Lemma 2. *For any $z_0 = (x_0, u_0, v_0) \in \mathcal{M}$, $I^+(z_0) \subseteq M \times (u_0, \infty) \times \mathbb{R}$ (resp. $I^-(z_0) \subseteq M \times (-\infty, u_0) \times \mathbb{R}$).*

Lemma 3. *Let $z_0 = (x_0, u_0, v_0), z_1 = (x_1, u_1, v_1), \Delta u = u_1 - u_0$. Any causal curve in \mathcal{M} with endpoints z_0, z_1 and velocity not proportional to ∂_v at any point, satisfies $|\Delta u| \neq 0$ and can be uniquely reparametrized as $\gamma(s) = (x(s), u(s), v(s)), \forall s \in I = [0, |\Delta u|], \gamma(0) = z_0$, in such a way that $\gamma(s)$ satisfies:*

³As Causality is conformal invariant, this also corresponds to both, the standard static case, and the case of GRW (Generalized Robertson-Walker spaces). Nevertheless, the construction can be carried out in the much more general setting of splitting type spacetimes (which include all the globally hyperbolic spacetimes [7]); see [47] for a general detailed study, or [48, section 3] for the case GRW.

(a) The u -component is written as:

$$u(s)(\equiv u_\nu(s)) := u_0 + \nu s, \quad \forall s \in I \quad (4.1)$$

where $\nu = \frac{\Delta u}{|\Delta u|}$ i.e., $\nu = 1$ when γ is future-directed and $\nu = -1$ when past-directed,

(b) putting $E(s) = \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle_L$ ($E(s) \leq 0, \forall s \in I$), then

$$v(s) = v_0 + \frac{\nu}{2} \int_0^s (-E(\sigma) + |\dot{x}(\sigma)|^2 - F(x(\sigma), u_\nu(\sigma))) d\sigma, \quad \forall s \in I. \quad (4.2)$$

Now, given any $(x_0, u_0), (x_1, u_1) \in M \times \mathbb{R}$, put $\Delta u = u_1 - u_0$ and assume $|\Delta u| \neq 0$. For each piecewise smooth curve $y : [0, |\Delta u|] \rightarrow M$ with endpoints x_0, x_1 , consider the unique lightlike curve $z(s) = (y(s), u_\nu(s), v_y(s)), s \in I = [0, |\Delta u|]$, u_ν as in (4.1), where $v_y(s)$ is determined from (4.2) (and thus, depends implicitly on ν) by putting $E(s) \equiv 0, v_0 = 0, x \equiv y$. So, if $\mathcal{C}(\equiv \mathcal{C}(x_0, x_1; |\Delta u|))$ denotes the set of all such curves y , a functional

$$\mathcal{C} \rightarrow \mathbb{R}, \quad y \mapsto v_y(|\Delta u|)$$

is obtained. In fact, define functional $\mathcal{J}_{u_0}^{\Delta u} : \mathcal{C} \rightarrow \mathbb{R}$:

$$\mathcal{J}_{u_0}^{\Delta u}(y) = \frac{1}{2} \int_0^{|\Delta u|} (|\dot{y}(s)|^2 - F(y(s), u_\nu(s))) ds. \quad (4.3)$$

Notice that, from the expression (4.2) for the component $v_y(s)$ we have:

$$v_y(|\Delta u|) = \nu \mathcal{J}_{u_0}^{\Delta u}(y).$$

Now, consider the *arrival map* $V : (M \times \mathbb{R}) \times (M \times \mathbb{R}) \rightarrow [-\infty, \infty]$,

$$((x_0, u_0), (x_1, u_1)) \mapsto V((x_0, u_0), (x_1, u_1)) := \text{Inf}_{\mathcal{C}} \mathcal{J}_{u_0}^{\Delta u} \in [-\infty, \infty) \quad (4.4)$$

($\Delta u = u_1 - u_0$; for convenience, $V = \infty$ if $u_0 = u_1$), which satisfies the triangle inequality

$$V((x_0, u_0), (x_2, u_2)) \leq V((x_0, u_0), (x_1, u_1)) + V((x_1, u_1), (x_2, u_2)), \quad (4.5)$$

whenever $u_0 < u_1 < u_2$ or $u_0 > u_1 > u_2$. Even more, from the expression (4.3) it directly follows that V is symmetric, i.e.:

$$V((x_0, u_0), (x_1, u_1)) = V((x_1, u_1), (x_0, u_0)), \quad (4.6)$$

whenever $u_0 \neq u_1$. From the construction, the following result (which shows that this function plays a similar role to time arrival Fermat's one) holds.

Proposition 3. For every $z_0 = (x_0, u_0, v_0) \in \mathcal{M}, x_1 \in M, u_1 \in \mathbb{R} \setminus \{u_0\}$:

If $u_1 > u_0$ then $z_1 = (x_1, u_1, v_1) \notin I^-(z_0)$ and:

$$z_1 = (x_1, u_1, v_1) \in I^+(z_0) \Leftrightarrow v_1 - v_0 > V((x_0, u_0), (x_1, u_1)).$$

If $u_1 < u_0$ then $z_1 = (x_1, u_1, v_1) \notin I^+(z_0)$ and:

$$z_1 = (x_1, u_1, v_1) \in I^-(z_0) \Leftrightarrow v_1 - v_0 < -V((x_0, u_0), (x_1, u_1)).$$

Proof. Clearly, the second case follows from the first one⁴ and, within this case, the first assertion follows from lemma 2. Then:

(\Rightarrow) Consider a timelike connecting curve ρ , and construct the lightlike broken geodesic $\gamma(s) = (y(s), u(s), v(s))$ provided by proposition 2 with $X = \partial_v$. Now, the $y(s)$ part yields the non-strict inequality, which is sufficient as the equality cannot hold ($I^+(z_0)$ is open and, thus, the non-strict inequality would follow also for a smaller v_1).

(\Leftarrow) If $\Delta u > 0$ then $V((x_0, u_0), (x_1, u_1))$ is the infimum of all the $v_y(|\Delta u|)$ for lightlike curves (at no point tangent to ∂_v) joining $(x_0, u_0, 0)$ with the line $\{(x_1, u_1)\} \times \mathbb{R}$. Thus, for some sequence $\{\epsilon_m\}_m \searrow 0$, the point z_0 can be joined with $p_m := (x_1, u_1, v_0 + V((x_0, u_0), (x_1, u_1)) + \epsilon_m)$ by means of a future-directed lightlike curve, and, for m big enough, p_m can be joined with z_1 by means of a (future-directed) integral curve of ∂_v . Thus, $z_0 < p_m < z_1$ and, as the three points do not lie on an (unbroken) lightlike geodesic, $z_0 \ll z_1$. ■

Remark 2.

1. Proposition 3 also ensures that Mp-waves do not admit event horizons (according to the criterion suggested in [31, sections 2.2.4] and refined in [20, subsection 3.2]), as any event z_0 can be joined with any line $\{(x_1, u_1)\} \times \mathbb{R}$ by means of either a future-directed (if $u_0 < u_1$) or a past-directed (if $u_0 > u_1$) timelike curve. This is similar to the inexistence of horizons in any spacetime which is standard static (on the whole manifold).
2. Due to the nature of our problem the domain of functional $\mathcal{J}_{u_0}^{\Delta u}$ will be the set of piecewise smooth curves $\mathcal{C}(\equiv \mathcal{C}(x_0, x_1; |\Delta u|))$. In fact, we will be interested in the qualitative properties of the infimum of $\mathcal{J}_{u_0}^{\Delta u}$ when Δu tends to some Δ_∞ , but not in the existence of a curve minimizing $\mathcal{J}_{u_0}^{\Delta u}$. When such a curve becomes relevant, a typical technical step is to enlarge \mathcal{C} by including curves with a lower degree of smoothness (H^1 curves) —remarkably, this happens in the problem of geodesic connectedness, see [14]. But even in the case of considering curves in such a enlarged \mathcal{C} , the corresponding curves of the Mp-wave (constructed according to lemma 3) are still causal [14, appendix], in the continuous sense explained in subsection 3.1.

5. Conditions on function F and functional \mathcal{J}

In order to get more information about the causal cones of these spacetimes, some technical conditions on functional $\mathcal{J}_{u_0}^{\Delta u}$ become crucial. These conditions are satisfied under natural restrictions on the growth of F . So, let us define first such relevant types of growth.

Definition 1. Let M be a connected Riemannian manifold, and consider the chosen point $\bar{x} \in M$ in (2.2). A function $F : M \times \mathbb{R} \rightarrow \mathbb{R}$ will be said:

⁴In what follows, even though the results will be stated for both, past and future, the proofs will be done only for one of them if there is no possibility of confusion.

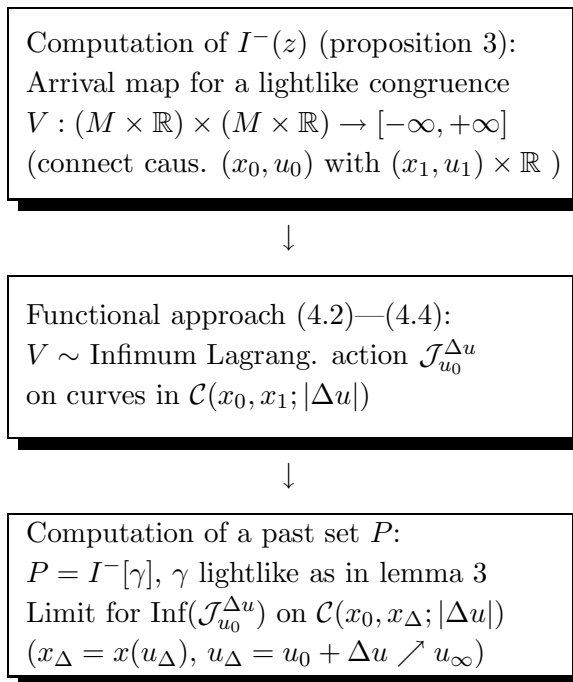


Figure 2: Emergency of the functional approach

- (i) superquadratic if M is unbounded and contains a sequence of points $\{p_m\}_m \subset M$ such that $|p_m| \rightarrow \infty$ and

$$R_1 \cdot |p_m|^{2+\epsilon} + R_0 \leq F(p_m, u) \quad \forall u \in \mathbb{R},$$

for some $\epsilon, R_1, R_0 \in \mathbb{R}$ with $\epsilon, R_1 > 0$.

- (ii) (spatially) at most quadratic if there exist continuous functions $R_0(u), R_1(u) > 0$ such that

$$F(x, u) \leq R_1(u)|x|^2 + R_0(u) \quad \forall (x, u) \in M \times \mathbb{R}. \quad (5.1)$$

Even more: (a) if (5.1) holds when $|x|^2$ is replaced by $|x|^{2-\epsilon(u)}$ for some continuous $\epsilon(u) > 0$, function F is called (spatially) subquadratic, and (b) if M is unbounded and a lower bound analogous to (5.1) also holds, i.e.,

$$R_1^-(u)|x|^2 + R_0^-(u) \leq F(x, u) \leq R_1(u)|x|^2 + R_0(u)$$

$R_1^-(u) > 0$ then F is (spatially) asymptotically quadratic.

- (iii) λ -asymptotically quadratic (on Mp-causal curves), with $\lambda > 0$, if M is unbounded and there exist continuous functions $R_0(u), R_1(u) > 0$ and a constant $R_0^- \in \mathbb{R}$ such that:

$$\frac{\lambda^2|x|^2 + R_0^-}{u^2 + 1} \leq F(x, u) \leq R_1(u)|x|^2 + R_0(u) \quad \forall (x, u) \in M \times \mathbb{R}.$$

Remark 3.

1. Of course, these definitions are independent of the choice of $\bar{x} \in M$ in (2.2). The exact value of functions R_0, R_1 is not relevant for the definitions and, thus, no more generality is gained if, say, a term in $|x|^{2-\epsilon(u)}$ is added to the right hand side of the inequalities in (ii) and (iii). Obviously:

$$\begin{aligned} \text{subquadratic} &\Rightarrow \text{at most quadratic} \Rightarrow \text{no superquadratic} \\ \lambda\text{-asymptotically quad.} &\Rightarrow \text{asymptotically quad.} \Rightarrow \text{at most quad.} \end{aligned}$$

2. For definitions (ii) the possible growth of F with u is essentially irrelevant (as R_i, R_i^- depend arbitrarily on u). Nevertheless, this is not the case for the lower bound (\leq) in (iii). The reason is that now the minimum quadratic behavior on F is required when computed on causal curves, i.e. for functions type $u \mapsto F(x(u), u)$. If λ, R_0^- depended arbitrarily on u , the inequality would be very weak. In principle, one would be forced to make the bound independent of u , i.e., type $\lambda^2|x|^2 + R_0^-$. Nevertheless, we allow a weakening of this bound just rescaling $|x|$ by dividing it by the same power of u , and even weaker conditions (as (5.9) below) would suffice.
3. Notice that conditions (i), (ii)(b) and (iii) impose restrictions on the minimal growth of F for large x and, thus, M is required to be unbounded. Nevertheless, condition (5.1) and (ii)(a) only bounds the upper growth of F and, so, if M is a bounded manifold, these definitions also make sense. In particular, any function F on a compact M will be regarded as subquadratic.
4. As proved by the authors in [19], if F is at most quadratic then the corresponding Mp-wave is strongly causal. Moreover, if the Riemannian manifold M is complete and F is subquadratic then the Mp-wave is globally hyperbolic. It is worth pointing out that Hubeny, Rangamani and Ross also studied stable causality by constructing explicitly time functions [33], and Minguzzi [41, theorem 5.5] related analytic properties of \mathcal{J} (in the more general framework of Eisenhart metrics) with the possible causal simplicity of the spacetime.

The following two technical conditions on $\mathcal{J}_{u_0}^{\Delta u}$ will be extensively used.

Definition 2. We will say that a Mp-wave \mathcal{M} satisfies hypothesis:

- (H1).** If, for each $u_0 \in \mathbb{R}$, there exists $\Delta_0 (= \Delta_0(u_0)) \geq 0$ such that for every $\Delta u > \Delta_0$ (resp. $\Delta u < -\Delta_0$), there exists a sequence of piecewise smooth loops $x_m : [0, |\Delta u|] \rightarrow M$ with the same base point $\bar{x} \in M$ (i.e., $x_m(0) = x_m(|\Delta u|) = \bar{x}$) satisfying

$$\mathcal{J}_{u_0}^{\Delta u}(x_m) \rightarrow -\infty \quad \text{when } m \rightarrow \infty. \tag{5.2}$$

- (H2).** If hypothesis **(H1)** holds with $\Delta_0 = 0$ for all u_0 .

Remark 4. Obviously, hypothesis **(H2)** implies **(H1)**, and there is no loss of generality assuming that the base point \bar{x} is equal to the point chosen in (2.2). Condition **(H1)** can be expressed in a simpler way, because if (5.2) holds for some $\Delta u = \Delta > 0$ then it also holds for all $\Delta u > \Delta$ (construct a piecewise smooth curve by “stopping” x_m during an interval of length $\Delta u - \Delta$).

In the next two lemmas, appropriate asymptotic behaviors of F are proved to be sufficient for these hypotheses.

Lemma 4. Hypothesis **(H2)** holds if F is superquadratic and $-F$ at most quadratic.

Proof. We will consider just the case $\Delta u > 0$. Choose $0 < \delta < \Delta u/2$ and take a sequence $\{p_m\}_m$ as in the definition of superquadratic. Let the sequence of curves $x_m : [0, \Delta u] \rightarrow M$ be defined as juxtapositions $x_m = \alpha_m^{-1} \star p_m \star \alpha_m$

$$x_m(s) = \begin{cases} \alpha_m(s) & \text{if } s \in [0, \delta] \\ p_m & \text{if } s \in [\delta, \Delta u - \delta] \\ \alpha_m(\Delta u - s) & \text{if } s \in [\Delta u - \delta, \Delta u], \end{cases} \quad (5.3)$$

where $\alpha_m : [0, \delta] \rightarrow M$ is a constant speed curve joining \bar{x} to p_m with length $L_m \leq |p_m| + 1$ for all m (if M were complete these curves could be chosen as minimizing geodesics of speed $L_m/\delta = |p_m|/\delta$). Clearly, the first term of $\mathcal{J}_{u_0}^{\Delta u}(x_m)$ in (4.3) satisfies the bound:

$$\int_0^{\Delta u} |\dot{x}_m(s)|^2 ds = \frac{2L_m^2}{\delta} \leq \frac{2(|p_m| + 1)^2}{\delta}. \quad (5.4)$$

And, from the hypotheses on F , the second term satisfies:

$$\begin{aligned} - \int_0^{\Delta u} F(x_m(s), u(s)) ds &= - \int_0^{\delta} F(x_m(s), u_0 + s) ds - \int_{\Delta u - \delta}^{\Delta u} F(x_m(s), u_0 + s) ds \\ &\quad - \int_{\delta}^{\Delta u - \delta} F(x_m(s), u_0 + s) ds \\ &\leq 2\delta(\tilde{R}_1 L_m^2 + \tilde{R}_0) - (\Delta u - 2\delta)(R_1 |p_m|^{2+\epsilon} + R_0) \\ &= -\bar{R}_1 |p_m|^{2+\epsilon} + (\text{terms in } |p_m|^2 \text{ and lower degree}), \end{aligned} \quad (5.5)$$

for some constants $R_1, R_0, \tilde{R}_1, \tilde{R}_0, \bar{R}_1 \in \mathbb{R}$, with $R_1, \bar{R}_1 > 0$. In conclusion, by adding (5.4) and (5.5) and recalling (4.3),

$$\mathcal{J}_{u_0}^{\Delta u}(x_m) \leq \frac{1}{\delta} (|p_m| + 1)^2 - \frac{1}{2} (\bar{R}_1 |p_m|^{2+\epsilon} - (\text{terms in lower degree})),$$

which clearly converges to $-\infty$ when $m \rightarrow \infty$, as required. ■

Lemma 5. Hypothesis **(H1)** holds if the Mp -wave satisfies any of the following conditions:

- (i) F is λ -asymptotically quadratic for some $\lambda > 1/2$.

(ii) $M = \mathbb{R}^n$ and F is the quadratic form

$$F(x, u) = \sum_{ij} f_{ij}(u)x^i x^j, \quad \text{with } f_{1j} \equiv f_{j1} \equiv 0 \text{ for all } j \neq 1,$$

$$f_{11}(u) \geq \lambda^2/(u^2 + 1) \quad \text{for large } |u| \text{ and some } \lambda > 1/2.$$

In particular, this includes the case $F(x, u) = \sum_{i=1}^n \mu_i (x^i)^2$ with $\mu_1 > 0$.

Proof. The very rough idea can be understood as follows. The loops x_m required for **(H1)** will be chosen by going and coming back from \bar{x} to an arbitrarily far point p_m , through a suitably parametrized (almost) geodesic x_m . Functional $\mathcal{J}_{u_0}^{\Delta u}(x_m)$ will be upper bounded essentially by

$$\int_0^{\Delta u} (\dot{y}^2 - R_1^- y^2) du \tag{5.6}$$

where $y(u) (\geq 0)$ represents the distance along x_m between \bar{x} and $x_m(u)$, and $R_1^-(u) \gtrsim \lambda u^{-a}$ for large u and $a \leq 2$. Recall that: (a) essentially, the contribution of the integrand of (5.6) is positive at the extremes (i.e., the base point of the loop), and negative around the maximum of $y(u)$, and (b) for **(H1)**, one only needs to study $\Delta u > \Delta_0$, so, one can try to find Δ_0 so big that the contribution of the negative term in (5.6) (say, with the curve staying a big time at p_m) is more important than the positive one. In fact, this is a good strategy when $a < 2$ but, in order to obtain an optimal bound when $a = 2$, the relative contributions of the negative and positive parts of (5.6) are delicate and depend heavily on the parametrization of the curve. So, we will consider the Euler-Lagrange equation for this functional, that is:

$$\ddot{y} = -R_1^- y,$$

with $y(0) = 0$. This is a concave function which, under our hypothesis, oscillates (in the sense of Sturm-Liouville theory) and, so, satisfies $y(\Delta u) = 0$ for some $\Delta u > 0$. This will yield good candidates to extremize the functional and, then, to obtain arbitrarily large negative values for it. These ideas will underlie in the following formal proof.

For case (i), let $p_m \in M$ be any sequence with $\{|p_m|\}_m \rightarrow \infty$, and $\alpha_m : [0, 1] \rightarrow M$ a sequence of constant speed curves joining \bar{x} to p_m , whose lengths L_m satisfy $L_m - |p_m| \searrow 0$ fast so that

$$(0 \leq) \quad (L_m s)^2 - |\alpha_m(s)|^2 \leq \nu_0 \quad \forall s \in [0, 1] \tag{5.7}$$

for some small $\nu_0 \geq 0$ (if M were complete, each α_m would be taken as a minimizing geodesic and (5.7) would hold for $\nu_0 = 0$). For some $0 < \epsilon < 1$ such that still $\epsilon \lambda > 1/2$, let $y_\epsilon(s)$ be the solution of the problem:

$$\begin{cases} \ddot{y}_\epsilon(s) = -R_1^-(s/\epsilon + u_0)y_\epsilon(s) & \text{with } R_1^-(u) = \lambda^2/(u^2 + 1) \\ \dot{y}_\epsilon(0) = 1 \\ y_\epsilon(0) = 0. \end{cases} \tag{5.8}$$

It is known from the very beginning of Sturm-Liouville theory that the lower bound

$$\limsup_{s \rightarrow \infty} [s^2 R_1^-(s/\epsilon + u_0)] = \epsilon^2 \lambda^2 > 1/4 \tag{5.9}$$

is the critical one for the existence of oscillatory solutions of (5.8), see [52, Ch. 6.3]. Therefore, inequality (5.9) ensures the existence of some $\Delta_0^* > 0$ (which may depend on ϵ) such that $y_\epsilon(\Delta_0^*) = 0$ (see [28, theorem 9] as a precise result).

From (5.8), obviously

$$(\dot{y}_\epsilon y_\epsilon) = \dot{y}_\epsilon^2 - R_1^-(s/\epsilon + u_0)y_\epsilon^2$$

and integrating:

$$\int_0^{\Delta_0^*} \dot{y}_\epsilon(s)^2 ds - \int_0^{\Delta_0^*} R_1^-(s/\epsilon + u_0) \cdot y_\epsilon(s)^2 ds = \dot{y}_\epsilon(\Delta_0^*)y_\epsilon(\Delta_0^*) - \dot{y}_\epsilon(0)y_\epsilon(0) = 0. \quad (5.10)$$

Now, for the chosen $\epsilon \in (0, 1)$, put

$$\Delta u := \Delta_0^*/\epsilon \quad z(s) := y_\epsilon(\epsilon \cdot s),$$

and notice:

$$\begin{aligned} \int_0^{\Delta u} \dot{z}(s)^2 ds - \int_0^{\Delta u} R_1^-(s + u_0) \cdot z(s)^2 ds \\ = \epsilon \int_0^{\Delta_0^*} \dot{y}_\epsilon(s)^2 ds - \frac{1}{\epsilon} \int_0^{\Delta_0^*} R_1^-(s/\epsilon + u_0) \cdot y_\epsilon(s)^2 ds < 0, \end{aligned}$$

the last inequality clearly from (5.10). In conclusion, the sequence of curves

$$x_m(s) := \alpha_m(z(s)/z_{\max}), \quad z_{\max} := \max\{z(s) : s \in [0, \Delta u]\}$$

will do the job for Δu , i.e.:

$$\begin{aligned} 2\mathcal{J}_{u_0}^{\Delta u}(x_m) &= \int_0^{\Delta u} |\dot{x}_m(s)|^2 ds - \int_0^{\Delta u} F(x_m(s), u(s)) ds \\ &\leq \int_0^{\Delta u} |\dot{x}_m(s)|^2 ds - \int_0^{\Delta u} (R_1^-(s + u_0)|x_m(s)|^2 + R_0^-(s + u_0)) ds \\ &\leq \frac{L_m^2}{z_{\max}^2} \left(\int_0^{\Delta u} \dot{z}(s)^2 ds - \int_0^{\Delta u} R_1^-(s + u_0) \cdot z(s)^2 ds \right) \\ &\quad - \int_0^{\Delta u} R_0^-(s + u_0) ds + \nu_0 \int_0^{\Delta u} R_1^-(s + u_0) ds \\ &\rightarrow -\infty, \end{aligned}$$

the last limit because $L_m \rightarrow \infty$ and the term in parentheses is negative. Notice that this divergence is shown for $\Delta u = \Delta_0^*/\epsilon$, which is sufficient according to remark 4.

Finally, for (ii) repeat the same reasoning but taking instead the sequence of loops $x_m(s) = (x_m^1(s), 0, \dots, 0)$ with $x_m^1(s) = L_m \cdot z(s)/z_{\max}$ (here $z(s)$ is derived analogously but using the lower bound for f_{11} instead of R_1^-). ■

Remark 5. *Relevant types of plane waves and pp-waves satisfy some of the sufficient conditions in lemmas 4, 5. Moreover, the behavior of F under condition (i) of lemma 5 is quite general and the estimates optimal. Nevertheless, we have not tried to give a more general (but probably less simple and transparent) result. In fact, this case (i) does not include the case (ii), which is completely independent. Roughly, condition **(H1)** holds*

$$\left. \begin{array}{l} \text{Superquadratic } F \\ + \\ \text{At most quadr. } - F \end{array} \right\} \xrightarrow{\text{lem 4}} (H2) \sim \left(\begin{array}{l} I^+(z_0) \text{ contains} \\ \text{region } u > u_0 \end{array} \right) \xrightarrow{\text{theorem 1}} \text{Non-distinguishing}$$

$$\left. \begin{array}{l} \lambda - \text{Asymp quad.} \\ \text{with } \lambda > 1/2 \\ \text{or} \\ \text{analogous condit.} \\ \text{in some direction} \\ \text{or} \\ \text{weaker Sturm} \\ \text{condit. as (5.9)} \end{array} \right\} \xrightarrow{\text{lem 5}} (h1) \sim \left(\begin{array}{l} I^+(z_0) \text{ contains} \\ \text{reg. } u > u_0 + \Delta_0 \end{array} \right) \xrightarrow{\text{figure 4}} \left(\begin{array}{l} P, \uparrow P \text{ explicit} \\ \partial\mathcal{M} \text{ low dim} \end{array} \right)$$

Figure 3: Consequences of the behaviour of F : technical conditions **(H1)**, **(H2)** (def. 2) vs asymptotic conditions (def. 1). The ($\lambda \leq 1/2$)-asymptotic case becomes critical (section 9.1) and the subquadratic case globally hyp. with expected higher dimension of $\partial\mathcal{M}$ (at least in the case M complete, sections 9.2, 9.3.)

when the system corresponding to (5.8) admits two zeroes. In particular, this happens when F behaves at least quadratically $\sim \lambda^2(|x|/u)^2$, $\lambda > 1/2$ (or just satisfying (5.9)) on the (x, u) part of a sequence of causal curves in \mathcal{M} with unbounded component x . So, a direction in the M part with this behaviour (where $|x|/u$ can be regarded as a sort of “rescaled distance”) suffices, see also remark 3 (2). This turns out the key behavior for the 1-dimensional character of the causal boundary.

On the other hand, by using Sturm-Liouville theory one can find conditions subtler than “ λ -asymptotically quadratic with $\lambda > 1/2$ ” (or directly (5.9)) in order to obtain the required oscillatory behavior for (5.8) and, thus, **(H1)** (see for example [28, theorem 10], [52, Ch. 6.3]). Nevertheless, in the natural types of asymptotic behaviors considered here, our estimates (for λ , powers of the distance and dependence on u) are the optimal ones, as shown in the explicit counterexample of subsection 9.1.

6. Non-distinguishing Mp-waves

In this section previous results are applied in order to prove that, when F is superquadratic, the causal structure of Mp-waves may become “degenerate” in certain sense. More precisely, such Mp-waves will not be distinguishing. As this is the minimum hypothesis in order to identify the points of \mathcal{M} with pairs (P, F) , these Mp-waves cannot admit a causal boundary. Nevertheless, this does not mean that these spacetimes may not be useful from the AdS/CFT viewpoint [34].⁵

⁵figure 1 in this reference may also help to understand the geometric situation.

6.1 The general result

Theorem 1. *A Mp-wave satisfying condition (H2) is neither future nor past-distinguishing. More concretely, under this hypothesis*

$$\begin{aligned} I^+(z_0) &= M \times (u_0, \infty) \times \mathbb{R} & \forall z_0 \in \mathcal{M} \\ I^-(z_0) &= M \times (-\infty, u_0) \times \mathbb{R} & \forall z_0 \in \mathcal{M}. \end{aligned}$$

In particular, this happens if F is superquadratic and $-F$ at most quadratic.

Proof. From lemma 2, to show

$$M \times (u_0, \infty) \times \mathbb{R} \subseteq I^+(z_0)$$

suffices, and by proposition 3, it is enough to check

$$\text{Inf}_{\mathcal{C}} \mathcal{J}(= V((x_0, u_0), (x_1, u_1))) = -\infty \quad \text{when } u_1 > u_0. \quad (6.1)$$

Thus, put $\Delta u = u_1 - u_0$ and choose $0 < \delta < \Delta u/2$. From (H2) there exists a sequence $x_m : [\delta, \Delta u - \delta] \rightarrow M$ satisfying the corresponding divergence (5.2). So, if $\alpha : [0, \delta] \rightarrow M$ and $\beta : [\Delta u - \delta, \Delta u] \rightarrow M$ are two fixed smooth curves joining x_0 to \bar{x} and \bar{x} to x_1 , respectively, the sequence of juxtaposed curves (as in (5.3)) $\{\beta \star x_m \star \alpha\}_m$ satisfies the required divergence for (6.1). ■

Remark 6. *If F is lower bounded then $-F$ is at most quadratic trivially. Thus, theorem 1 extends our previous result [19, proposition 2.1]. On the other hand, theorem 1 can be extended clearly to obtain the cases future and past distinguishing independently (split condition (H2) in future and past cases in an obvious way).*

As it is well-known, plane waves are always strongly causal, and thus, cannot lie under the hypotheses of previous theorem. However, this result is useful to decide if many other pp-waves of possible interest to string theorists can admit a causal boundary.

6.2 Some remarkable examples

Essentially, the following examples are taken from Hubeny and Rangamani [30]. The expectations to obtain a 1-dimensional boundary are truncated here, as the pp-waves may be non-distinguishing—a possibility already suggested by the own authors and Ross in [33].

- (1) Consider the pp-wave $\mathcal{M} = \mathbb{R}^n \times \mathbb{R}^2$ with

$$F(x^1, \dots, x^n, u) = \cosh x^1 - \cos x^2. \quad (6.2)$$

This spacetime leads to the $\mathcal{N} = 2$ sine-Gordon theory on the world-sheet in light-cone quantization. \mathcal{M} does not admit a causal boundary, since function F in (6.2) is bounded below and superquadratic (take for example $p_m = (m, 0, \dots, 0)$ in definition 1 (i)), and so, theorem 1 (or previous computations in [19, 33]) applies.

(2) Consider the generalization of previous case to a pp-wave with

$$F(x^i, u) = \sum_j f_j(x^j).$$

In [30] the authors studied the case of a single coordinate $F(x, u) = f(x)$. They stated that the causal boundary is 1-dimensional whenever $f(x)$ is bounded from below and, in addition, $f(x \rightarrow \pm\infty) \rightarrow +\infty$. This agrees with our results if $f \sim x^2$ at infinity. However, from theorem 1, these conditions lead to non-distinguishing spacetimes whenever f (or one of the functions f_j) behaves superquadratically (for example, $F(x, u) = x^4$) and thus, the boundary is not well defined.

(3) Another examples in [30] are the 4-dimensional vacuum pp-wave spacetime with $F((x^1, x^2), u) = -\sin x^1 e^{x^2}$ or the 5-dimensional pp-wave $\mathcal{M} = \mathbb{R}^3 \times \mathbb{R}^2$ with

$$F(r, \theta, \phi, u) = r^3 (5 \cos^3 \theta - 3 \cos \theta).$$

In these cases, function F is superquadratic but $-F$ is not at most quadratic. However, condition **(H2)** still holds because both conditions on F hold in at least one direction, say $x_1 = -\pi/2$ for the first example, or $\theta = 0$ for the second one (explicitly, take, say, $x_m(s) = (m \sin \frac{\pi}{\Delta u} s, 0, 0)$ in the second example). Thus, the causal boundary is not well defined again by theorem 1.

(4) Finally, consider an arbitrary 4-dimensional vacuum pp-wave spacetime; i.e.,

$$\mathcal{M} = \mathbb{R}^4, \quad \langle \cdot, \cdot \rangle_L = d(x^1)^2 + d(x^2)^2 - F(x, u) du^2 - 2 du dv,$$

with function $F(x, u)$ spatially harmonic ($\partial_{x^1}^2 F + \partial_{x^2}^2 F \equiv 0$). As pointed out in [20], in this case there are only three possibilities:⁶

- (a) either F is superquadratic, and thus, the causal boundary makes no sense in general,⁷ or
- (b) $F(x^1, x^2, u) = f(u) \left((x^1)^2 - (x^2)^2 \right) + 2g(u)x^1x^2$, and then we have a plane wave (see subsection 8.3), or
- (c) $F(x, u) = a(u) + b(u)x^1 + c(u)x^2$. In this last case the pp-wave is Lorentz-Minkowski space,⁸ and thus, the causal boundary is the classical double cone (also for the new concept of causal boundary [18, example 10.1]).

⁶For each constant u , either the harmonic function $F(\cdot, u)$ is superquadratic or it is polynomially bounded, and thus, it becomes a polynomial (of at most the degree of the bound; in this case, 2): the result is well-known for holomorphic functions; for harmonic ones stronger results can be seen, for example, at [35, lemma 4.1].

⁷Moreover, in this case the pp-wave would be incomplete, according to a conjecture by Ehlers and Kundt [17].

⁸Notice that the curvature vanishes (see for example [13, section 2], [20, formula (3)]) and the spacetime is complete [13, proposition 3.5] and simply connected.

7. Boundaries in strongly causal Mp-waves

From now on, the ambient hypothesis on \mathcal{M} will be strong causality, so that \mathcal{M} admits a causal boundary with a natural topology. Nevertheless, we will state it explicitly because most of the computations in the next subsection are valid for any Mp-wave.

From corollary 1, only (right) inextendible ν -lightlike curves

$$\gamma : [0, \nu\Delta_\infty) \rightarrow \mathcal{M}, \quad \nu\Delta_\infty \in (0, \infty], \quad (7.1)$$

with $\dot{\gamma}(s)$ independent of $X = \partial_v$ for all s , are needed in order to compute the pairs $(P, F) \in \partial\mathcal{M}$. Here again $\nu = \pm 1$ keeps track of the causal orientation of γ ($\nu = 1$ for future-directed γ and $\nu = -1$ for past directed) and, so, $I^{-\nu}[\gamma]$, $\uparrow^\nu \gamma$ will denote $I^-[\gamma]$, $\uparrow \gamma$ (resp. $I^+[\gamma]$, $\downarrow \gamma$) if $\nu = 1$ (resp. $\nu = -1$); for simplicity, the reader can consider the case $\nu = 1$ and check the final expressions for $\nu = -1$. *In what follows, we will work under a reparametrization $\gamma(s) = (x(s), u_\nu(s), v(s))$ as in lemma 3; notice that $v(s)$ is given by (4.2) with $E(s) \equiv 0$.*

We also put $\gamma(0) = z_0 = (x_0, u_0, v_0)$ and $u_\infty = u_0 + \Delta_\infty \in [-\infty, \infty]$. For any $\nu\Delta \in (0, \nu\Delta_\infty)$, we will consider the restriction $\gamma|_{[0, \nu\Delta]}$ and put $\gamma(\nu\Delta) = z_\Delta = (x_\Delta, u_\Delta, v_\Delta)$.

Remark 7. *Recall that the curve γ must be inextendible. As γ is reconstructed from its spatial part, x will be said inextendible when: (i) $\nu\Delta_\infty = \infty$ (i.e., γ is inextendible in u), (ii) $\nu\Delta_\infty < \infty$ but x is not continuously extendible to $\nu\Delta_\infty$ (γ is inextendible in x), or (iii) $\nu\Delta_\infty < \infty$, x is continuously extendible to $\nu\Delta_\infty$ but the (total kinetic) energy diverges, i.e.: $(1/2) \int_0^{\nu\Delta_\infty} |\dot{x}(s)|^2 ds = \infty$ (γ is inextendible in v).*

7.1 General expressions for $P, F, \uparrow P, \downarrow F$

Let us start with $I^{-\nu}[\gamma]$. From proposition 3, a point $\bar{z}_0 = (\bar{x}_0, \bar{u}_0, \bar{v}_0) \in \mathcal{M}$ with $\nu\bar{u}_0 < \nu u_\infty$ lies in $I^{-\nu}[\gamma]$ if and only if (recall the symmetry of V , see (4.6)),

$$\nu(v_\Delta - \bar{v}_0) > V((\bar{x}_0, \bar{u}_0), (x_\Delta, u_\Delta)) \quad (7.2)$$

for some $\nu\Delta > 0$ (close to $\nu\Delta_\infty$). Put

$$\begin{aligned} V_\Delta &= \nu(v_\Delta - v_0), \\ V_\Delta(\bar{x}_0, \bar{u}_0) &= V((\bar{x}_0, \bar{u}_0), (x_\Delta, u_\Delta)) (= V((x_\Delta, u_\Delta), (\bar{x}_0, \bar{u}_0))), \end{aligned} \quad (7.3)$$

that is,

$$\begin{aligned} V_\Delta &= \frac{1}{2} \int_0^{|\Delta|} (|\dot{x}(s)|^2 - F(x(s), u_\nu(s))) ds \\ V_\Delta(\bar{x}_0, \bar{u}_0) &= \inf_{\mathcal{C}} \mathcal{J}_{\bar{u}_0}^{\bar{\Delta}} = \inf_{y \in \mathcal{C}} \left\{ \frac{1}{2} \int_0^{|\bar{\Delta}|} (|\dot{y}(s)|^2 - F(y(s), \bar{u}_0 + \bar{\nu}s)) ds \right\}, \end{aligned} \quad (7.4)$$

where

$$\bar{\Delta} := u_\Delta - \bar{u}_0 \quad (7.5)$$

(here $\mathcal{C} \equiv \mathcal{C}(\bar{x}_0, x_\Delta; |\bar{\Delta}|)$ is the set of piecewise smooth curves defined in $[0, |\bar{\Delta}|]$ joining \bar{x}_0 with x_Δ , and $\bar{\nu} = \bar{\Delta}/|\bar{\Delta}|$; recall also that, by hypothesis on \bar{z}_0 , $\nu = \bar{\nu}$ for Δ close to Δ_∞). Now condition (7.2) translates into

$$V_\Delta - V_\Delta(\bar{x}_0, \bar{u}_0) > \nu(\bar{v}_0 - v_0) \quad \text{for } \Delta \text{ close to } \Delta_\infty. \quad (7.6)$$

Lemma 6. *The left-hand side of (7.6) is non-decreasing when $\nu\Delta \nearrow \nu\Delta_\infty$.*

Proof. Close to $\nu\Delta_\infty$ and for small $\nu\epsilon > 0$, we have $\nu\bar{u}_0 < \nu u_\Delta < \nu u_{\Delta+\epsilon} (< \nu u_\infty)$, and by using the triangle inequality (4.5):

$$V_{\Delta+\epsilon} = V_\Delta + \frac{1}{2} \int_{|\Delta|}^{|\Delta+\epsilon|} (|\dot{x}(s)|^2 - F(x(s), u_\nu(s))) ds$$

$$V_{\Delta+\epsilon}(\bar{x}_0, \bar{u}_0) \leq V_\Delta(\bar{x}_0, \bar{u}_0) + \inf_{y \in \mathcal{C}'} \frac{1}{2} \int_{|\bar{\Delta}|}^{|\bar{\Delta}+\epsilon|} (|\dot{y}(s)|^2 - F(y(s), \bar{u}_0 + \nu s)) ds,$$

where now \mathcal{C}' is equal to $\mathcal{C}(x_\Delta, x_{\Delta+\epsilon}; \nu\epsilon)$ up to the reparametrization (by means of a translation) with domain $[|\bar{\Delta}|, |\bar{\Delta} + \epsilon|]$. Thus, as claimed,

$$V_{\Delta+\epsilon} - V_{\Delta+\epsilon}(\bar{x}_0, \bar{u}_0) \geq V_\Delta - V_\Delta(\bar{x}_0, \bar{u}_0).$$

■

Thus, taking the limit $\nu\Delta \nearrow \nu\Delta_\infty$ in (7.6), the following result is obtained.

Proposition 4. *Let γ be a inextendible ν -lightlike curve (as in (7.1)). For each $\bar{z}_0 = (\bar{x}_0, \bar{u}_0, \bar{v}_0)$, put:*

$$b^-(\bar{x}_0, \bar{u}_0) = \lim_{\nu\Delta \nearrow \nu\Delta_\infty} (V_\Delta - V_\Delta(\bar{x}_0, \bar{u}_0)) \quad (7.7)$$

(with $V_\Delta, V_\Delta(\bar{x}_0, \bar{u}_0)$ defined in (7.3)). Then:

$$I^{-\nu}[\gamma] = \{\bar{z}_0 \in \mathcal{M} : \nu\bar{u}_0 < \nu u_\infty \text{ and } b^-(\bar{x}_0, \bar{u}_0) > \nu(\bar{v}_0 - v_0)\}.$$

Next, let us consider the common future (or past) $\uparrow^\nu \gamma$ for γ . From proposition 3, a point $\bar{z}_0 = (\bar{x}_0, \bar{u}_0, \bar{v}_0) \in \mathcal{M}$ with $\nu\bar{u}_0 \geq \nu u_\infty$ lies in $I^{+\nu}[\gamma(\nu\Delta)]$ if and only if (recall the notation in (7.3))

$$\nu(\bar{v}_0 - v_\Delta) > V_\Delta(\bar{x}_0, \bar{u}_0)$$

that is,

$$V_\Delta + V_\Delta(\bar{x}_0, \bar{u}_0) < \nu(\bar{v}_0 - v_0). \quad (7.8)$$

Reasoning as in lemma 6, the triangle inequality (4.5) implies that the left-hand side of (7.8) is non-decreasing with $\nu\Delta$ (but now apply it taking into account $\nu\bar{u}_0 > \nu u_{\Delta+\epsilon} > \nu u_\Delta$, for $\nu\epsilon > 0$). So, the non-strict inequality will hold in (7.8) when the limit $\nu\Delta \nearrow \nu\Delta_\infty$ is taken. This will be the key for the following result.

Proposition 5. *Let γ be a inextendible ν -lightlike curve (as in (7.1)). For each $\bar{z}_0 = (\bar{x}_0, \bar{u}_0, \bar{v}_0)$, put:*

$$b^+(\bar{x}_0, \bar{u}_0) = \lim_{\nu\Delta \nearrow \nu\Delta_\infty} (V_\Delta + V_\Delta(\bar{x}_0, \bar{u}_0)) \quad (7.9)$$

(with $V_\Delta, V_\Delta(\bar{x}_0, \bar{u}_0)$ defined in (7.3)). Then:

$$\uparrow^\nu \gamma = I^{+\nu}[\{\bar{z}_0 \in \mathcal{M} : \nu \bar{u}_0 \geq \nu u_\infty \text{ and } b^+(\bar{x}_0, \bar{u}_0) \leq \nu(\bar{v}_0 - v_0)\}]$$

Proof. (For $\nu = 1$). The inclusion \subseteq for $\uparrow \gamma$ follows easily from the reasoning above.

For the converse, let $\bar{z}'_0 \gg \bar{z}_0$, with \bar{z}_0 such that $\bar{u}_0 \geq u_\infty$ and $b^+(\bar{x}_0, \bar{u}_0) \leq \bar{v}_0 - v_0$. We can choose $\bar{z}'_0 \gg \bar{z}''_0 \gg \bar{z}_0$ and we only need to show $\bar{z}''_0 \gg \gamma(\Delta)$ for all Δ . Since $V_\Delta + V_\Delta(\bar{x}_0, \bar{u}_0)$ is non-decreasing, the condition on $b^+(\bar{x}_0, \bar{u}_0)$ implies

$$V_\Delta + V_\Delta(\bar{x}_0, \bar{u}_0) \leq \bar{v}_0 - v_0, \quad \text{for all } \Delta. \quad (7.10)$$

On the other hand, condition $\bar{z}''_0 \gg \bar{z}_0$ implies

$$V((\bar{x}_0, \bar{u}_0), (\bar{x}''_0, \bar{u}''_0)) < \bar{v}''_0 - \bar{v}_0. \quad (7.11)$$

Thus, adding (7.10), (7.11) and using the triangle inequality (4.5):

$$V_\Delta + V_\Delta(\bar{x}''_0, \bar{u}''_0) < \bar{v}''_0 - v_0,$$

that is,

$$V_\Delta(\bar{x}''_0, \bar{u}''_0) < \bar{v}''_0 - v_\Delta,$$

as required. ■

Recall that, by using lemma 3, the lightlike curve γ in previous two propositions can be reconstructed from its initial point $\gamma(0)$, its x -part and its future or past causal character $\nu = \pm 1$; in particular, functions b^\pm can be constructed from u_0, ν and curve $x(s)$. Nevertheless, in order to obtain the sets $\uparrow^\nu I^{-\nu}[\gamma]$ associated to each $I^{-\nu}[\gamma]$ by means of these propositions, one must take into account that technicalities appear when γ (necessarily a lightlike pregeodesic) is not included in $I^{-\nu}[\gamma]$ (in fact, here perhaps $\uparrow^\nu \gamma \neq \uparrow^\nu I^{-\nu}[\gamma]$; recall remark 1 and corollary 1). Fortunately, the following lemma shows that this situation cannot happen in our case.

Lemma 7. *Let $\gamma : [0, |\Delta_\infty|) \rightarrow \mathcal{M}$ be a inextendible ν -lightlike curve constructed from lemma 3. Then there exists an inextendible ν -timelike curve $\rho : [0, |\Delta_\infty|) \rightarrow \mathcal{M}$ such that $I^{-\nu}[\gamma] = I^{-\nu}[\rho]$ and $\uparrow^\nu \gamma = \uparrow^\nu \rho$.*

Proof. Construct ρ from γ as follows. Take some negative function $E(s) < 0$ with $-\int_0^{|\Delta_\infty|} E(s) ds = \epsilon \in (0, \infty)$. Then, ρ will have the same parts $u(s), x(s)$ of γ , but compute the $v(s)$ part from (4.2) using the chosen function $E(s)$ and replacing v_0 by $v_0 - \nu\epsilon$. Obviously, $I^{-\nu}[\gamma] \supseteq I^{-\nu}[\rho]$ and $\uparrow^\nu \gamma \subseteq \uparrow^\nu \rho$. For the converses, remake the proofs of propositions 4, 5 for ρ , checking that the additional term in $E(s)$ does not affect to the limits for b^\pm . ■

Summing up, this subtlety plus propositions 4, 5 yields the following characterization of TIP's and TIF's.

Theorem 2. *Any TIP, P (resp. TIF, F) of a strongly causal Mp-wave (2.1) is constructed as follows. Take $(u_0, v_0) \in \mathbb{R}^2$, a piecewise smooth curve $x : [0, |\Delta_\infty|) \rightarrow M$ inextendible to*

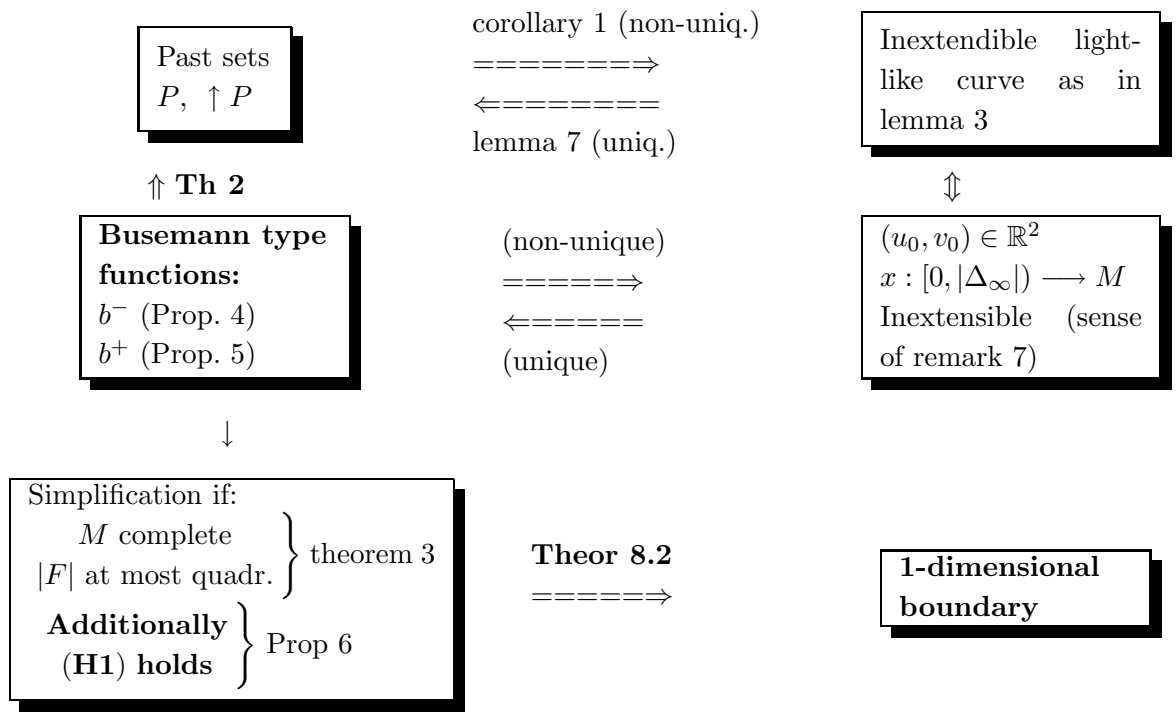


Figure 4: Computation of $P, \uparrow P$ in strongly causal Mp-waves. The scheme of general computation in terms of Busemann type functions is summarized in the four upper boxes. In the two bottom ones, under some mild technical simplifications, hypothesis **(H1)** implies the 1-dimensional boundary. In boldface crucial conclusions, beyond the technical development.

$|\Delta_\infty|$ (in the sense of remark 7) and the function b^- associated to u_0, x and $\nu = 1$ (resp. $\nu = -1$) from lemma 3 and (7.7). Putting $\Delta_\infty = \nu|\Delta_\infty|$ and $u_\infty = u_0 + \Delta_\infty$ one has:

$$P = \{\bar{z}_0 \in \mathcal{M} : \bar{u}_0 < u_\infty \text{ and } b^-(\bar{x}_0, \bar{u}_0) > \bar{v}_0 - v_0\}$$

(resp. $F = \{\bar{z}_0 \in \mathcal{M} : \bar{u}_0 > u_\infty \text{ and } b^-(\bar{x}_0, \bar{u}_0) > v_0 - \bar{v}_0\}$).

Even more, taking also the function b^+ from (7.9):

$$\uparrow P = I^+[\{\bar{z}_0 \in \mathcal{M} : \bar{u}_0 \geq u_\infty \text{ and } b^+(\bar{x}_0, \bar{u}_0) \leq \bar{v}_0 - v_0\}]$$

(resp. $\downarrow F = I^-[\{\bar{z}_0 \in \mathcal{M} : \bar{u}_0 \leq u_\infty \text{ and } b^+(\bar{x}_0, \bar{u}_0) \leq v_0 - \bar{v}_0\}]$).

Proof. Let P be a TIP. By corollary 1, P can be written as the chronological past of a lightlike curve γ as in (7.1). Applying propositions 4, 5 to γ the required expressions for $P, \uparrow P$ holds.

Conversely, let P be a set defined as in the expression above, and take the associated inextendible future-directed lightlike curve γ such that $P = I^-[\gamma]$. By lemma 7, P is a TIP and $\uparrow P = \uparrow \gamma$, as required. ■

7.2 Boundary for M complete, $|F|$ at most quadratic

In order to get a manageable causal boundary we need to impose not only strong causality but also a pair of (natural and not too restrictive) hypotheses more. The first one is the completeness of the Riemannian part M . Otherwise, the Riemannian Cauchy boundary of M (i.e., the boundary for the completion as a metric space by using Cauchy sequences) can complicate the causal boundary. Technically, completeness yields the following well-known property, to be used later. If the curve $x : [0, |\Delta_\infty|) \rightarrow M, |\Delta_\infty| < \infty$ is not continuously extendible to $|\Delta_\infty|$ (i.e., it lies in the case (ii) of remark 7) then the completeness of M implies that its length diverges and, by Cauchy-Schwarz inequality, so does its energy, i.e.:

$$\int_0^{|\Delta_\infty|} |\dot{x}(s)|^2 ds = \infty. \tag{7.12}$$

The second one is that not only F must be at most quadratic (which is the natural sufficient bound for strong causality [19, theorem 3.1]), but also so must be $|F|$. Otherwise, other interesting geometric properties of the spacetime, as the geodesic completeness of the whole Mp-wave, may be destroyed (even in the simplest case of M complete and F independent of u), see remark 8.

Under these two hypotheses, we will obtain a technical property for $\uparrow \gamma$ or $\downarrow \gamma$ (lemma 8(ii)), plus a remarkable simplification for TIP's and TIF's, namely, *any TIP determined by a $(\nu = 1)$ -lightlike curve with $|\Delta_\infty| < \infty$, is just the region $u < u_\infty$* (lemma 8(i)).

Remark 8. *This simplification is also pointed out in [30, section 5.1]. Nevertheless, the hypothesis $|F|$ at most quadratic is missing there. The following example shows that it cannot be dropped. Consider the (3-dimensional) pp-wave $\mathcal{M} = \mathbb{R}^3$ with $F(x, u) = -x^4$. This is globally hyperbolic (as F is subquadratic) and incomplete. In fact, the future-directed lightlike geodesic $\gamma : [0, u_\infty) \rightarrow \mathcal{M}$, $\gamma(s) = (y(s), u(s), v(s))$ determined by*

$$s(y) = \int_0^y \frac{1}{\sqrt{1 + \bar{y}^4}} d\bar{y}, \quad u(s) = s, \quad v(0) = 0,$$

is incomplete, as u_∞ (the integral until $y = \infty$) is finite. Obviously, $I^-[\gamma] \subset \{\bar{z}_0 : \bar{u}_0 < u_\infty\}$ but the inclusion is strict. In fact, $y(s)$ strictly minimizes functional⁹ $\int_0^{s_0} (\dot{x}(s)^2 + x(s)^4) ds$ for all $s_0 \in [0, u_\infty)$. From (4.4) the arrival function V satisfies:

$$V((0, 0), (y(u), u)) = \frac{1}{2} \int_0^u (\dot{y}(\sigma)^2 + y(\sigma)^4) d\sigma = v(u), \quad \forall u \in (0, u_\infty).$$

So, from the interpretation of V (Prop. 3), $(0, 0, v_0) \notin I^-(\gamma(u))$ for any $u \in (0, u_\infty)$, whenever $v_0 \geq 0$.

⁹This can be checked from general arguments: (a) looking at the functional as a Lagrangian with negative "potential" $V = -x^4/2$, a minimum must be attained for each $s_0 \in (0, u_\infty)$, (b) this minimum must be attained by a solution of the corresponding Euler-Lagrange equation $\ddot{x} - 2x^3 = 0$ (one of its solutions being $y(s)$), and (c) the boundary conditions $x(0) = y(0) = 0, x(s_0) = y(s_0)$, determine univocally the solution (recall that, from standard theory of equations, a second solution $x(s)$ would be fixed univocally by $x(0), \dot{x}(0)$, but, for example, $\dot{x}(0) > \dot{y}(0) \Rightarrow x(s_0) > y(s_0)$).

In order to avoid these difficulties, from now on we will assume that $|F|$ is at most quadratic and M complete:

Lemma 8. *Under these two hypotheses, if the inextendible causal curve $\gamma : [0, \nu\Delta_\infty) \rightarrow \mathcal{M}$ satisfies $\nu\Delta_\infty < \infty$ then:*

- (i) $b^-(\bar{x}_0, \bar{u}_0) = \infty$ if $\nu\bar{u}_0 < \nu u_\infty$,
- (ii) *there exists $\nu\delta > 0$ such that $b^+(\bar{x}_0, \bar{u}_0) = \infty$ whenever $\nu u_\infty \leq \nu\bar{u}_0 < \nu(u_\infty + \delta)$. Even more, $|\delta| = \infty$ if F is subquadratic.*

Proof. Since γ is inextendible, so is its component x , thus, its energy (7.12) diverges. From the at most quadratic behaviour of $|F|$, and the fact that the image of u_ν lies in a compact subset, we have, up to an additive constant:

$$2V_\Delta \geq A_\Delta := \int_0^\Delta |\dot{x}(s)|^2 ds - R \int_0^\Delta |x(s)|^2 ds, \quad \text{for some } R > 0 \quad (7.13)$$

(recall the first formula in (7.4)).

- (i) As $V_\Delta(\bar{x}_0, \bar{u}_0)$ is obtained taking an infimum in $\mathcal{J}_{\bar{u}_0}^{\bar{\Delta}}$ (recall $\bar{\Delta} = u_\Delta - \bar{u}_0 = \Delta + u_0 - \bar{u}_0$), from Prop. 4 it is enough to exhibit a curve $y_\Delta \in \mathcal{C}(\bar{x}_0, x_\Delta; |\bar{\Delta}|)$ for each Δ close to Δ_∞ , such that

$$\lim_{\Delta \nearrow \Delta_\infty} V_\Delta - \mathcal{J}_{\bar{u}_0}^{\bar{\Delta}}(y_\Delta) = \infty. \quad (7.14)$$

Concretely, y_Δ will be taken as a minimizing geodesic. In fact, for some constants $C_1, C_2 > 0$ (and assuming $\bar{x}_0 = \bar{x}$ in (2.2) without loss of generality)

$$2\mathcal{J}_{\bar{u}_0}^{\bar{\Delta}}(y_\Delta) = \frac{|x_\Delta|^2}{|\bar{\Delta}|} - \int_0^{|\bar{\Delta}|} F(y_\Delta(s), \bar{u}_0 + \nu s) ds \leq C_1|x_\Delta|^2 + C_2,$$

the equality by taking into account the minimizing character of y_Δ , and the inequality by the at most quadratic bound of $|F|$ and the fact that $|\bar{\Delta}|$ is bounded ($|\Delta_\infty| < \infty$). Therefore, the mentioned inequality $2V_\Delta \geq A_\Delta$ plus corollary 2 (its last assertion) yields the required limit (7.14).

- (ii) As we have seen V_Δ diverges and, thus, it is enough to prove the existence of $\nu\delta > 0$ such that $V_\Delta(\bar{x}_0, \bar{u}_0)$ is lower bounded for any $\nu\Delta \in (0, \nu\delta)$. From the at most quadratic behaviour of F we have

$$2\mathcal{J}_{\bar{u}_0}^{\bar{\Delta}}(y) \geq \int_0^{|\bar{\Delta}|} |\dot{y}(s)|^2 ds - \int_0^{|\bar{\Delta}|} (R_1|y(s)|^2 + R_0) ds$$

for any $y \in \mathcal{C}(\bar{x}_0, x_\Delta; |\bar{\Delta}|)$ and for Δ such that $\nu\Delta \geq \nu(\Delta_\infty - 1)$ (so that the coefficients $R_1(u), R_0(u)$ for at most quadraticity can be replaced by their maximums in u). Then, the required δ and lower bound are straightforward from proposition 8. ■

Notice that part (i) of lemma 8 plus proposition 4 yield:

$$I^{-\nu}[\gamma] = \{\bar{z}_0 \in \mathcal{M} : \nu\bar{u}_0 < \nu u_\infty (< \infty)\},$$

and the part (ii) joined to proposition 5 yield:

$$\uparrow^\nu \gamma \subset \{\bar{z}_0 \in \mathcal{M} : \nu\bar{u}_0 > \nu(u_\infty + \delta)\},$$

which is an information additional to theorem 2.

Summarizing, the two ambient hypotheses of lemma 8 yield the following strengthening of the conclusions of theorem 2.

Theorem 3. *Let \mathcal{M} be a Mp-wave with $|F|$ at most quadratic and M complete. Choosing $(u_0, v_0) \in \mathbb{R}^2$, $x : [0, |\Delta_\infty|) \rightarrow M$ and b^- as in theorem 2, the equalities for non-empty past and future sets read as:*

$$P = \begin{cases} \bar{z}_0 : \bar{u}_0 < u_\infty & \text{if } u_\infty < \infty \\ \bar{z}_0 : b^-(\bar{x}_0, \bar{u}_0) > \bar{v}_0 - v_0 & \text{if } u_\infty = \infty, \end{cases}$$

$$F = \begin{cases} \bar{z}_0 : \bar{u}_0 > u_\infty & \text{if } u_\infty > -\infty \\ \bar{z}_0 : b^-(\bar{x}_0, \bar{u}_0) > v_0 - \bar{v}_0 & \text{if } u_\infty = -\infty. \end{cases}$$

Even more, for each P, F as above there exists $\nu\delta > 0$ such that:

$$\uparrow P = I^+[\{\bar{z}_0 : \bar{u}_0 \geq u_\infty + \nu\delta \text{ and } b^+(\bar{x}_0, \bar{u}_0) \leq \bar{v}_0 - v_0\} \subset \{\bar{z}_0 : \bar{u}_0 > u_\infty + \nu\delta\},$$

$$(\text{resp. } \downarrow F = I^-[\{\bar{z}_0 : \bar{u}_0 \leq u_\infty - \nu\delta \text{ and } b^+(\bar{x}_0, \bar{u}_0) \leq v_0 - \bar{v}_0\} \subset \{\bar{z}_0 : \bar{u}_0 < u_\infty - \nu\delta\}),$$

and, if F is subquadratic, one can take $\delta = \infty$, i.e.:

$$\uparrow P = \emptyset, \quad \downarrow F = \emptyset.$$

Remark 9. *Notice that, in order to write the pairs $(P, F) \in \partial\mathcal{M}$, a TIF F cannot be S -related with two TIP's P, P' . In fact, the corresponding u_∞ should be finite for P and P' (otherwise, the common future would be empty) and, thus, one of them, say P , would be included in the other, P' (contradicting the maximality of P in $\downarrow F$).*

In the subquadratic case for $|F|$, $\partial\mathcal{M}$ is the union of all the pairs (P, \emptyset) and (\emptyset, F) ; in particular, no ideal points in $\hat{\partial}\mathcal{M}$ and $\check{\partial}\mathcal{M}$ are identified (this is a general fact, for globally hyperbolic spacetimes [18, theorem 9.1]). In the general at most quadratic case, pairs (P, F) with none of the components empty are allowed (as well as identifications between points in $\hat{\partial}\mathcal{M}$ and $\check{\partial}\mathcal{M}$). But, even in this case, a non-empty P can form an ideal point with at most one F , and viceversa.

8. Mp-waves with natural 1-dim. $\partial\mathcal{M}$

8.1 Collapsing to i^\pm

We begin by studying the case of lightlike curves with diverging component u .

Proposition 6. *Let \mathcal{M} be a Mp-wave with $|F|$ at most quadratic, M complete, and satisfying condition **(H1)** in Def. 2. If $\gamma : [0, \infty) \rightarrow \mathcal{M}$ is a ν -lightlike curve, then*

$$I^{-\nu}[\gamma] = \mathcal{M}, \quad \uparrow^\nu \gamma = \emptyset.$$

Proof. Clearly, the second equality directly follows from theorem 3. For the first one, fix $\bar{x}_0 \in \mathcal{M}$. Again from theorem 3, it suffices to show that $b^-(\bar{x}_0, \bar{u}_0) = \infty$. To this aim, we only need to prove (recall (7.7)):

$$V_\Delta(\bar{x}_0, \bar{u}_0) = \left(\inf_{\mathcal{C}} \mathcal{J}_{\bar{u}_0}^{\bar{\Delta}} \right) - \infty \quad \text{for all } \Delta \text{ big enough} \quad (8.1)$$

with $\bar{\Delta} = \Delta + u_0 - \bar{u}_0$, and $\mathcal{C} \equiv \mathcal{C}(\bar{x}_0, x_\Delta; |\bar{\Delta}|)$. Choose all the Δ 's such that $\bar{\Delta} - 2$ is greater than the value of $\Delta_0 = \Delta_0(\bar{u}_0 + 1)$ given by hypothesis **(H1)**. Consider the following constant speed smooth curves: $\alpha : [0, 1] \rightarrow M$ joining \bar{x}_0 to \bar{x} and $\beta_\Delta : [\bar{\Delta} - 1, \bar{\Delta}] \rightarrow M$ connecting \bar{x} to $x(\Delta)$. Let x_m be the sequence of piecewise smooth loops provided by hypothesis **(H1)** for $u_0 = \bar{u}_0 + 1$ and $\Delta u = \bar{\Delta} - 2$. The sequence of juxtaposed curves $y_{\Delta_m} = \beta_\Delta \star x_m \star \alpha$, i.e.,

$$y_{\Delta_m}(s) = \begin{cases} \alpha(s) & \text{if } s \in [0, 1] \\ x_m(s-1) & \text{if } s \in [1, \bar{\Delta} - 1] \\ \beta_\Delta(s) & \text{if } s \in [\bar{\Delta} - 1, \bar{\Delta}], \end{cases}$$

satisfies:

$$\begin{aligned} \mathcal{J}_{\bar{u}_0}^{\bar{\Delta}}(y_{\Delta_m}) &= \frac{1}{2} \int_0^{\bar{\Delta}} |\dot{y}_{\Delta_m}(s)|^2 ds - \frac{1}{2} \int_0^{\bar{\Delta}} F(y_{\Delta_m}(s), \bar{u}_0 + s) ds \\ &= \frac{1}{2} \text{length}(\alpha)^2 + \frac{1}{2} \text{length}(\beta_\Delta)^2 \\ &\quad - \frac{1}{2} \int_0^1 F(\alpha(s), \bar{u}_0 + s) ds - \frac{1}{2} \int_{\bar{\Delta}-1}^{\bar{\Delta}} F(\beta_\Delta(s), \bar{u}_0 + s) ds \\ &\quad + \mathcal{J}_{\bar{u}_0+1}^{\bar{\Delta}-2}(x_m). \end{aligned}$$

Thus, hypothesis **(H1)** ensures that $\mathcal{J}_{\bar{u}_0}^{\bar{\Delta}}(y_{\Delta_m})$ goes to $-\infty$ when m goes to $+\infty$, and (8.1) holds, as required. ■

With this result and theorem 3 at hand, our aim in the next subsections is to formalize precisely the cases when the boundary of the wave is a lightlike line.

8.2 Case asymptotically quadratic

Now, if we take into account the boundary construction in subsection 3.2, we can establish the following result:

Theorem 4. *The causal boundary $\partial\mathcal{M}$ of a Mp-wave with F λ -asymptotically quadratic for some $\lambda > 1/2$, and M complete has the following structure:*

1. *As a point set, two copies L^+, L^- of \mathbb{R} , with eventual identifications between the points of the copies, plus two ideal points i^+, i^- . In fact, $\partial\mathcal{M}$ will be written as a union (non-necessarily disjoint, due to the identifications) $\partial\mathcal{M} = \hat{\partial}\mathcal{M} \cup \check{\partial}\mathcal{M}$ where $\hat{\partial}\mathcal{M} \equiv L^+ \cup \{i^+\}$ and $\check{\partial}\mathcal{M} \equiv \{i^-\} \cup L^-$.*

2. Topologically, the following natural homeomorphisms hold: $\hat{\partial}\mathcal{M} \cong (-\infty, \infty]$, $\check{\partial}\mathcal{M} \cong [-\infty, \infty)$. Moreover, $\partial\mathcal{M}$ is a quotient topological space with the possible identifications allowed in (a) above.
3. Causally, $\hat{\partial}\mathcal{M}$, $\check{\partial}\mathcal{M}$, with the restriction of the weak causal relation in $\partial\mathcal{M}$, are totally ordered and weakly locally lightlike (i.e., each Q in, say, $\hat{\partial}\mathcal{M}$ has a neighbourhood $\mathcal{L} \subseteq \hat{\partial}\mathcal{M}$ such that: any $Q_1, Q_2 \in \mathcal{L}$ are weakly horismotically related in $\hat{\partial}\mathcal{M}$ if and only if $Q_1 < Q_2$ as points of $(-\infty, \infty]$).

Proof. From lemma 5 (i) and definition 1 (iii), the hypotheses of theorem 3, proposition 6 hold. Therefore, directly from proposition 6 and theorem 3:

$$\begin{aligned}
 I^-[\gamma] = \mathcal{M}, \quad \uparrow \gamma = \emptyset & \quad \text{if } \Delta_\infty = \infty, \\
 \left\{ \begin{array}{l} I^-[\gamma] = \{\bar{z}_0 : \bar{u}_0 < u_\infty\} \\ \uparrow \gamma = I^+[\{\bar{z}_0 : \bar{u}_0 \geq u_\infty + \delta, b^+(\bar{x}_0, \bar{u}_0) + v_0 - \bar{v}_0 \leq 0\}] \end{array} \right. & \quad \text{if } \Delta_\infty < \infty, \quad (8.2)
 \end{aligned}$$

for any future-directed lightlike curve γ with $u(s) = u_0 + s$. Thus, the future causal boundary $\hat{\partial}\mathcal{M}$ contains the ideal point i^+ and a copy L^+ corresponding to the line $u_\infty \in (-\infty, \infty)$. Moreover, the chronological topology clearly attaches i^+ to the right extreme of L^+ (and it is the natural topology on L^+). On the other hand, any two points $u_\infty, u'_\infty \in L^+$, $u_\infty < u'_\infty$, are weakly causally related, since the corresponding pairs of terminal sets (P, F) , (P', F') satisfy:

$$P = \{\bar{z}_0 : \bar{u}_0 < u_\infty\} \subset \{\bar{z}_0 : \bar{u}_0 < u'_\infty\} = P'. \quad (8.3)$$

Moreover, taking into account that $F \subset \uparrow P \subset \{\bar{z}_0 : \bar{u}_0 > u_\infty + \delta\}$ for some $\delta > 0$ (recall theorem 3), one has, for $u_\infty < u'_\infty \leq u_\infty + \delta$,

$$F \cap P' \subset \{\bar{z}_0 : u_\infty + \delta < \bar{u}_0 < u'_\infty\} = \emptyset.$$

Whence, (P, F) , (P', F') are not chronologically related, and $\hat{\partial}\mathcal{M}$ is weakly locally lightlike.

Analogously, the past causal boundary $\check{\partial}\mathcal{M}$ can be represented by another copy L^- of the line $u_\infty \in (-\infty, \infty)$ plus the ideal point i^- attached at the left extreme, and is weakly locally lightlike.

Finally, the (total) causal boundary $\partial\mathcal{M}$ is formed by $L^+ \cup \{i^+\} \cup L^- \cup \{i^-\}$, up to eventual identifications between those ideal points in L^- , L^+ represented by the same pair of terminal sets, and all the conclusions follow. ■

Remark 10. Notice that we have stated only the weak causal relation, as we have proven $P \subset P'$ in (8.3) but not $F' \subset F$. The possible difficulty for this inclusion appears only in the very particular case that F' is a maximal TIF into $\uparrow P$, and P' a maximal TIP into $\downarrow F'$, and thus $F = \emptyset$. This situation cannot happen if, for example, $F(x, u)$ is independent of u , since then F is maximal TIF into $\uparrow P$ if and only if P is maximal TIP into $\downarrow F$. As a consequence, the boundary in this case becomes locally lightlike for the natural causal relation.

8.3 Plane waves

Consider now the case of a plane wave $\mathcal{M} = \mathbb{R}^n \times \mathbb{R}^2$,

$$F(x, u) = \sum_{i,j} f_{ij}(u)x^i x^j, \quad f_{ij} = f_{ji}.$$

For simplicity, assume that F has the form of lemma 5 (ii) and, thus, falls under the hypotheses of theorem 3 and proposition 6. Then, reasoning as in theorem 4:

Theorem 5. *The causal boundary of a plane wave with $f_{1j} \equiv 0$ for all $j \neq 1$, and $f_{11}(u) \geq \lambda^2/(u^2 + 1)$, for large $|u|$ and some $\lambda > 1/2$, is as described in theorem 4, remark 10.*

Remark 11. *Some particular cases where f_{ij} is diagonal have been computed by Hubeny and Rangamani in [30], and it is worth comparing here. They used the existence of “oscillating geodesics” as an evidence of a 1-dimensional boundary. The items in [30, subsection 4.3] labelled 1, 2, NL1, NL3 as well as the case $f_{11}(u) = 1/(u^2 + 1)$ of item 4 (or the singular case NL2) do have such oscillating geodesics, and are particular cases of theorem 5. The case $f_{11}(u) = \cos u$ (item 3), is included in the technique, as it satisfies trivially the inequality (5.9) and, thus the conclusion of lemma 5 holds (see remark 5). In the singular case $f(u) = \lambda^2/u^2$ (item 6) they obtain oscillatory geodesics for $\lambda^2 > 1/4$, also in agreement with theorem 5. As shown in subsection 9.1 by means of a counterexample, one cannot expect a 1-dimensional boundary even in the limit case $\lambda^2 = 1/4$. So, it is not surprising now that, if $f_{11}(u) = e^{-u^2}$ (as in [30, subsection 4.3, item 5]) the oscillatory behaviour ceases.*

Very roughly, in our approach the infimum of some functional is considered, and in Hubeny and Rangamani’s just the (lightlike geodesics associated to the) critical curves of this functional. Of course, when the infimum is attained the minimizing curve is critical, but our functional approach has clear advantages. In fact, it relies only on the qualitative functional properties rather than on the exact details of the Euler-Lagrange equation. The oscillating geodesics in the most accurate Hubeny and Rangamani’s results, imply the existence of a solution with two zeros for the Euler-Lagrange equation of our simplified functional (5.6) (see the discussion around this formula), and this is enough for the results.

Recall that only the 1-dimensional character of the boundary is ensured by theorems 5, 4. The question of establishing which ideal points in L^+ and L^- must be identified becomes hard, and depends on the behaviour of function b^+ in (8.2). The only additional information on b^+ is provided by lemma 8(ii) (or, equivalently, by the expressions of $\uparrow P, \downarrow F$ in theorem 3).

Nevertheless, identifications can be easily computed in the highly symmetric case of plane waves with $F(x, u)$ independent of¹⁰ u , i.e., $F(x, u) = \sum_{i,j} \mu_{ij} x^i x^j$, with μ_{ij} symmetric coefficient matrix. Here, each $\uparrow P$ is equal to some F and viceversa [36]. As remarked in [36], these Mp-waves contain many interesting examples for string theory (maximally supersymmetric 11-dimensional solution obtained from the Penrose limit of $AdS_4 \times S^7$ and

¹⁰They are usually called *homogeneous* plane waves, even though the name *locally symmetric* is intrinsic and seems more appropriate, see for example [23].

$AdS_7 \times S^4$ [9], partially supersymmetric plane waves in ten dimensions [16, 4, 40], including the Penrose limit of the Pilch-Warner flow [15, 25, 10]). Due to the exceptionality of this case, we will not attempt a very general result here. Simply, we will give an extended version of the result in [36], in order to check how our technique works. More general results would rely on the possibility to reformulate lemma 9 below and extend formulas (8.4), (8.6).

Concretely, now we assume that function f_{11} in theorem 5 is constant and equal to the biggest eigenvalue μ_1 of the matrix $f_{ij}(u)$, and $\mu_1 > 0$.

Lemma 9. *Under these hypotheses, let $\gamma : [0, |\Delta_\infty|) \rightarrow \mathcal{M}$ be an inextendible ν -lightlike curve, with $|\Delta_\infty| \in (0, \infty)$, $\nu\Delta_\infty > 0$ and $u_\infty := u_0 + \Delta_\infty$. Then:*

$$\begin{aligned} \text{If } \nu = 1, & & \uparrow \gamma = \mathbb{R}^n \times (u_\infty + \pi/\mu_1, \infty) \times \mathbb{R}. \\ \text{If } \nu = -1, & & \downarrow \gamma = \mathbb{R}^n \times (-\infty, u_\infty - \pi/\mu_1) \times \mathbb{R}. \end{aligned}$$

Proof. (For $\nu = 1$.) \supseteq . Clearly, if $\bar{z}'_0 \in \mathbb{R}^n \times (u_\infty + \pi/\mu_1, \infty) \times \mathbb{R}$ then $\bar{z}'_0 \gg \bar{z}_0$ for some $\bar{z}_0 = (\bar{x}_0, \bar{u}_0, \bar{v}_0)$ with $\bar{u}_0 = u_\infty + \pi/\mu_1$. Therefore, from proposition 5 the required inclusion follows by proving $b^+(\bar{x}_0, \bar{u}_0 = u_\infty + \pi/\mu_1) = -\infty$, or just (recall (7.9)):

$$V_\Delta(\bar{x}_0, \bar{u}_0) = -\infty \quad \text{for all } \Delta < \Delta_\infty \text{ close to } \Delta_\infty. \quad (8.4)$$

Thus, for Δ close to Δ_∞ , consider $|\bar{\Delta}| (> \pi/\mu_1)$ as in (7.5) and take $0 < \delta_\Delta < |\bar{\Delta}|/2$ small enough such that

$$\mu_1^2 \geq \frac{\pi^2 + \epsilon_\Delta}{(|\bar{\Delta}| - 2\delta_\Delta)^2}, \quad \text{for some } \epsilon_\Delta > 0. \quad (8.5)$$

Define the juxtapositions

$$y_{\Delta_m}(s) = \begin{cases} -\frac{x(\Delta)}{\delta_\Delta}s + x(\Delta) & \text{if } s \in [0, \delta_\Delta] \\ (y_{\Delta_m}^1(s), 0, \dots, 0) & \text{if } s \in [\delta_\Delta, |\bar{\Delta}| - \delta_\Delta] \\ \frac{\bar{x}_0}{\delta_\Delta}s + \frac{\delta_\Delta \bar{x}_0 - |\bar{\Delta}| \bar{x}_0}{\delta_\Delta} & \text{if } s \in [|\bar{\Delta}| - \delta_\Delta, |\bar{\Delta}|], \end{cases}$$

with

$$y_{\Delta_m}^1(s) = m \sin\left(\frac{\pi}{|\bar{\Delta}| - 2\delta_\Delta}(s - \delta_\Delta)\right) \quad \forall s \in [\delta_\Delta, |\bar{\Delta}| - \delta_\Delta].$$

Then, from (8.5) we obtain

$$\begin{aligned} \mathcal{J}_{u_\Delta}^{\bar{\Delta}}(y_{\Delta_m}) &= \frac{1}{2} \int_0^{|\bar{\Delta}|} (|\dot{y}_{\Delta_m}(s)|^2 - F(y_{\Delta_m}(s), u_\Delta + s)) ds \\ &= \frac{1}{2} \left(\int_{\delta_\Delta}^{|\bar{\Delta}| - \delta_\Delta} |\dot{y}_{\Delta_m}^1(s)|^2 ds - \mu_1^2 \int_{\delta_\Delta}^{|\bar{\Delta}| - \delta_\Delta} y_{\Delta_m}^1(s)^2 ds \right) + \Lambda_\Delta \\ &\leq \frac{1}{2} \int_{\delta_\Delta}^{|\bar{\Delta}| - \delta_\Delta} |\dot{y}_{\Delta_m}^1(s)|^2 ds - \frac{\pi^2 + \epsilon_\Delta}{2(|\bar{\Delta}| - 2\delta_\Delta)^2} \int_{\delta_\Delta}^{|\bar{\Delta}| - \delta_\Delta} y_{\Delta_m}^1(s)^2 ds + \Lambda_\Delta \\ &= -\frac{\epsilon_\Delta m^2}{4(|\bar{\Delta}| - 2\delta_\Delta)} + \Lambda_\Delta \end{aligned}$$

for some $\Lambda_\Delta \in \mathbb{R}$ independent of m . Summing up, $\mathcal{J}_{u_\Delta}^{\bar{\Delta}}(y_{\Delta_m}) \rightarrow -\infty$ when $m \rightarrow +\infty$, and (8.4) holds.

⊆. We will prove that, if $\bar{z}'_0 \notin \mathbb{R}^n \times (u_\infty + \pi/\mu_1, \infty) \times \mathbb{R}$ then $\bar{z}_0 \notin \cap_\Delta I^+[\gamma(\Delta)]$ for any $\bar{z}_0 \ll \bar{z}'_0$ (and thus, $\bar{z}'_0 \notin \uparrow \gamma$). From lemma 2, $\bar{u}_0 - u_\infty < \pi/\mu_1$, and by proposition 5, it is enough:

$$V_\Delta(\bar{x}_0, \bar{u}_0) > -\infty \quad \text{is lower bounded for all } \Delta < \Delta_\infty \text{ close to } \Delta_\infty \quad (8.6)$$

(recall (7.9) and the fact that $V_\Delta \rightarrow \infty$ because of (7.13) and corollary 2). From the hypotheses, $|\bar{\Delta}| \leq (\pi - \epsilon_0)/\mu_1$, for some $\epsilon_0 > 0$, and for all $\Delta < \Delta_\infty$ close enough. Therefore,

$$\begin{aligned} \mathcal{J}_{u_\Delta}^{\bar{\Delta}}(y) &= \frac{1}{2} \int_0^{|\bar{\Delta}|} (|\dot{y}(s)|^2 - F(y(s), u_\Delta + s)) ds \\ &\geq \frac{1}{2} \left(\int_0^{|\bar{\Delta}|} |\dot{y}(s)|^2 ds - \mu_1^2 \int_0^{|\bar{\Delta}|} |y(s)|^2 ds \right) \\ &\geq \frac{1}{2|\bar{\Delta}|} \left(|\bar{\Delta}| \int_0^{|\bar{\Delta}|} |\dot{y}(s)|^2 ds - \frac{(\pi - \epsilon_0)^2}{|\bar{\Delta}|} \int_0^{|\bar{\Delta}|} |y(s)|^2 ds \right). \end{aligned}$$

As $V_\Delta(\bar{x}_0, \bar{u}_0)$ is obtained by taking the infimum in this expression, the bound for λ in theorem 7 (see appendix) ensures (8.6), as required. ■

Theorem 3 and lemma 9 tell us that the pair $(I^-[\gamma], \uparrow \gamma)$ with $u \nearrow u_\infty$ coincides with $(\downarrow \tilde{\gamma}, I^+[\tilde{\gamma}])$ with $u \searrow u_\infty + \pi/\mu_1$, i.e., each future ideal point represented by some $u_\infty \in L^+$ must be identified with the past ideal point represented by $u_\infty + \pi/\mu_1 \in L^-$ (and there are no more identifications). Summing up:

Theorem 6. *Let \mathcal{M} be a plane wave with $f_{1j} \equiv 0$ for all $j \neq 1$, and $f_{11}(u)$ a positive constant function equal to the biggest eigenvalue of $f_{ij}(u)$ (in particular, any locally symmetric plane wave with a positive eigenvalue). Then, $\partial\mathcal{M}$ is weakly locally lightlike and canonically identifiable to $[-\infty, \infty]$, both as a point set and as a topological space, being the weak causal relation the corresponding one to the natural order. Even more, in the locally symmetric case this also holds for the causal relation.*

9. Higher dimensionality of $\partial\mathcal{M}$

When F grows less fast than quadratic (in all directions) one does not expect a 1-dimensional boundary. In fact, if F is subquadratic and M complete then the Mp-wave becomes globally hyperbolic. So, there are no identifications between $\hat{\partial}\mathcal{M}$, $\check{\partial}\mathcal{M}$ and, the structure of the spacetime suggests a boundary with two pieces which resemble in some sense the Cauchy hypersurfaces¹¹ –notice that the Cauchy hypersurfaces are necessarily noncompact and, at least when M is non-compact, one could expect that some portion

¹¹If M were not complete, global hyperbolicity may be destroyed, but the main difference in the expected picture is that additional boundary points would appear, associated to inextendible curves in M with finite energy.

of $\partial\mathcal{M}$ were higher dimensional, even of dimension $(n + 1)$. Some concrete cases will be briefly analyzed in subsections 9.2, 9.3. But, first, we will see that the $(\lambda = 1/2)$ -asymptotic quadratic growth of F becomes critical for the 1-dimensional character of the boundary. Recall that this case appears in geometries derived from NS5 branes, see [30, section 4.3, NL2].

9.1 Criticality of $\lambda = 1/2$ for 1-dimensionality

Consider for simplicity a pp-wave $\mathcal{M} = \mathbb{R}^{n+2}$ with $F = F_\lambda, \lambda \in \mathbb{R}$, satisfying:

$$F_\lambda(x, u) = \lambda^2|x|^2/(1 + u)^2, \tag{9.1}$$

for $u \geq 0$ (and eventually for $u < -2$, but we will not take care of this part). Obviously, F_λ is λ -asymptotically quadratic and, for $\lambda > 1/2$, $\hat{\partial}\mathcal{M}$ is 1-dimensional (and so essentially $\partial\mathcal{M}$). Our purpose is to show that this is *not* the case for $\lambda = 1/2$, which shows the optimal character of our results.

Concretely, we will construct ν -lightlike curves $\gamma : [0, \infty) \rightarrow \mathcal{M}$ with $u(s) \nearrow \infty$ such that $I^-[\gamma] \neq \mathcal{M}$. Thus, the collapse of all the corresponding ideal points to the single one i^+ (which was essential in section 8 –proposition 6– in order to ensure the 1-dimensionality of the boundary) will not hold. As a technical previous step:

Lemma 10. *Let $F = F_{1/2}$ in (9.1) and $n = 1$. Consider the functional*

$$\mathcal{J}_0^{\Delta u}(x) = \int_0^{\Delta u} (\dot{x}^2 - F(x(u), u)) du$$

and the solution $y(u) = \sqrt{1 + u}$ to the Euler-Lagrange equation

$$\ddot{y} + p(u)y = 0 \quad p(u) = 1/4(1 + u)^2.$$

Then

$$\text{Inf}_{x \in \mathcal{C}(1, y(\Delta u); \Delta u)} \mathcal{J}_0^{\Delta u} = \mathcal{J}_0^{\Delta u}(y|_{[0, \Delta u]}) = 0$$

for all $\Delta u > 0$.

Proof. The last equality is straightforward, so, we will see that $y|_{[0, \Delta u]}$ minimizes the functional by usual techniques from Sturm-Liouville theory (see [3, section 1.1], [52, Ch. 4]). Put $g(u) = \dot{y}(u)/y(u)$, which satisfies Riccati's equation $\dot{g} + g^2 = -p$. For any $x \in \mathcal{C}(1, y(\Delta u); \Delta u)$ one has:

$$\mathcal{J}_0^{\Delta u}(x) = \int_0^{\Delta u} (\dot{x}^2 - px^2) du = \int_0^{\Delta u} (\dot{x} - xg)^2 du + x^2(u)g(u) \Big|_{u=0}^{u=\Delta u} \tag{9.2}$$

(expand the first term in the right side and integrate by parts $2 \int x\dot{x}g = \int \dot{x}^2g$). And taking into account that curves x, y coincide at the extremes:

$$\mathcal{J}_0^{\Delta u}(x) \geq x^2(u)g(u) \Big|_{u=0}^{u=\Delta u} = y^2(u)g(u) \Big|_{u=0}^{u=\Delta u} = \mathcal{J}_0^{\Delta u}(y)$$

(the last equality applying (9.2) to $x = y$). ■

Now, consider the lightlike curve in the pp-wave $\gamma(u) = (x(u), u, v(u))$ constructed from lemma 3 with $x(u) = y(u)\vec{e}$, where \vec{e} is any unit vector of \mathbb{R}^n and $y(u) = \sqrt{1+u}$ (and $v(0) = 0$). From (4.4) and lemma 10, the arrival function V satisfies:

$$V((x(0), 0), (x(u), u)) = 0, \quad \forall u > 0.$$

Thus, from the interpretation of V (proposition 3), $z = (x(0), 0, v_0) \notin I^-(\gamma(u))$ whenever $v_0 \geq 0 (= v(u))$.

Remark 12. Notice that this not only proves the required inequality $I^-(\gamma) \neq \mathcal{M}$. In fact, moving \vec{e} in all the directions ($\vec{e} \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$), and $v(0) \in \mathbb{R}$, different curves $\gamma = \gamma[\vec{e}, v(0)]$ are obtained. Each one yields an ideal point, that is, a portion of $\partial\mathcal{M}$ containing a n -dimensional subset of ideal points is constructed.

9.2 Static and Minkowski type Mp-waves

According to García-Parrado and Senovilla [22], a spacetime \mathcal{M} is called *causally related* with a second one \mathcal{M}' , shortly $\mathcal{M} \prec \mathcal{M}'$, if a diffeomorphism ϕ maps the causal cones of \mathcal{M} into the ones of \mathcal{M}' ; moreover, $\mathcal{M}, \mathcal{M}'$ are *isocausal* if $\mathcal{M} \prec \mathcal{M}'$ and $\mathcal{M}' \prec \mathcal{M}$. Intuitively, when $\mathcal{M} \prec \mathcal{M}'$ the causal cones of \mathcal{M}' can be obtained by opening the ones of \mathcal{M} . If they are isocausal then they are not necessarily conformal, but many causal properties are shared by both spacetimes [22, 21].

When a Mp-wave has coefficient $F(x, u)$ bounded in x then it becomes isocausal to the simplest choice $F \equiv 0$, more precisely:

Proposition 7. Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ be a Mp-wave with $|F(x, u)| \leq f(u)$ for all $(x, u) \in M \times \mathbb{R}$, where f is a continuous function. Then $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is isocausal to the standard static spacetime obtained just making $F \equiv 0$, i.e.

$$\mathcal{M} = M \times \mathbb{R}^2, \quad g_0 = \langle \cdot, \cdot \rangle - 2dudv.$$

Proof. By a simple computation of the causal cones, the metrics

$$g^\pm := \langle \cdot, \cdot \rangle \pm f(u)du^2 - 2dudv$$

satisfy

$$(\mathcal{M}, g^-) \prec (\mathcal{M}, \langle \cdot, \cdot \rangle_L) \prec (\mathcal{M}, g^+).$$

But recall that both metrics g^\pm are isometric to the static (\mathcal{M}, g_0) , as shown by the global change of coordinates:

$$\tilde{u} = u, \quad \tilde{v} = v \mp \frac{1}{2} \int_0^u f(\sigma) d\sigma.$$

■

So, even though the relation between the causal boundaries of two isocausal spacetimes does not seem trivial, one expects that, when proposition 7 applies, the boundary of the Mp-wave will not be too different to the boundary of the corresponding static model. In particular, when $M = \mathbb{R}^2$ the static spacetime is \mathbb{L}^{n+2} , so, if pp-waves are considered, one expects a boundary not very different to Lorentz-Minkowski's.

9.3 The case $-F$ quadratic

Marolf and Ross [36] proved that the *conformal* boundary is a set of two lightlike hyperplanes joined by two lightlike lines, in the case of (conformally flat) locally symmetric plane waves with equal negative eigenvalues. Now, we will extend that proof to include non-locally symmetric ones. Then, the causal boundary will be also computed and, as we will see, the picture will be a bit different.

We will also focus on the simplest case of a (non-locally symmetric) plane wave with equal negative eigenvalues of F . This corresponds to the case $-F$ quadratic (which can be studied in further detail with the introduced techniques). Thus, let $\mathcal{M} = \mathbb{R}^{n+2}$ with

$$\langle \cdot, \cdot \rangle_L = dx^2 + |x|^2 f(u) du^2 - 2du dv, \quad f(u) > 0, \quad (9.3)$$

where, $x = (x^1, \dots, x^n)$. Consider the differential equation

$$\ddot{r}(u) = f(u)r(u), \quad r(0) = 1, \quad \dot{r}(0) = 0. \quad (9.4)$$

The change of variables

$$x = r(u)\tilde{x}, \quad v = \tilde{v} + \frac{1}{2}r(u)\dot{r}(u)\tilde{x}^2$$

takes (9.3) into

$$\langle \cdot, \cdot \rangle_L = r(u)^2 d\tilde{x}^2 - 2d\tilde{u}d\tilde{v},$$

on all \mathbb{R}^{n+2} . Thus, the further change of variable $\tilde{u} = \int_0^u \frac{du'}{r(u')^2}$ yields the explicitly conformally flat expression:

$$\langle \cdot, \cdot \rangle_L = r(\tilde{u})^2 (d\tilde{x}^2 - 2d\tilde{u}d\tilde{v}). \quad (9.5)$$

Observe that the domain for coordinate \tilde{u} is given by:

$$\tilde{u}_{-\infty} < \tilde{u} < \tilde{u}_{\infty} \quad \text{with} \quad \tilde{u}_{\pm\infty} := \int_0^{\pm\infty} \frac{ds}{r(s)^2}, \quad 0 < \pm\tilde{u}_{\pm\infty} < \infty,$$

being the finiteness of $\tilde{u}_{\pm\infty}$ a consequence of the convexity of r in (9.4). Therefore, the plane wave is conformal to the proper region $\tilde{u}_{-\infty} < \tilde{u} < \tilde{u}_{\infty}$ of Minkowski spacetime (in the coordinates of (9.5)). In particular, the conformal boundary (for the restriction of the classical Minkowski embedding) consists of two parallel lightlike hyperplanes at $\tilde{u} = \pm\tilde{u}_{\infty}$ and two lightlike lines (say, two copies of $[\tilde{u}_{-\infty}, \tilde{u}_{\infty}]$) which represent the intersection of the region $\tilde{u}_{-\infty} \leq \tilde{u} \leq \tilde{u}_{\infty}$ with the past and future infinity \mathcal{J}^{\pm} of Minkowski space.

Now, recall that the conformal version (9.5) of the plane wave (9.3) can be also used to compute the causal boundary, and it looks like somewhat different. In fact, this boundary contains again two lightlike hyperplanes (which can be identified in \mathbb{L}^{n+2} with pairs $(I^-(z), \emptyset) \in \hat{\partial}\mathcal{M}$, where $u(z) = \tilde{u}_{\infty}$, and $(\emptyset, I^+(z)) \in \check{\partial}\mathcal{M}$ with $u(z) = \tilde{u}_{-\infty}$), and two lightlike lines. But these lines are now identified naturally with copies $(\tilde{u}_{-\infty}, \tilde{u}_{\infty}] \subset \hat{\partial}\mathcal{M}$ and $[\tilde{u}_{-\infty}, \tilde{u}_{\infty}) \subset \check{\partial}\mathcal{M}$ (say, as no future-directed timelike curve approaches $\tilde{u}_{-\infty}$). Notice that, *both $\hat{\partial}\mathcal{M}$ and $\check{\partial}\mathcal{M}$ are connected and non-compact, and there are no identifications for $\partial\mathcal{M}$; thus, plainly $\partial\mathcal{M} = \hat{\partial}\mathcal{M} \cup \check{\partial}\mathcal{M}$.*

10. Conclusions

We have carried out a systematic study of Mp-waves, being our main goals:

1. We consider the very wide family of wave-type spacetimes (2.1) and determine the general qualitative behaviour of the metric which yields a 1-dimensional causal boundary, as well as other properties, see table 1.
2. Even though we particularize our general results to many cases, our main aim is to introduce general techniques potentially applicable to other cases of interest in General Relativity, String Theory or other theories. These techniques involve a functional approach, Sturm-Liouville theory, the introduction of new Busemann type functions and technicalities on Causality.
3. The functional approach (which is a variant of the one introduced in [19]) is also interpreted as an arrival time function, with clear analogues to Fermat's principle one. This interpretation also clarifies the causal structure of the waves, including the inexistence of horizons.
4. Our study includes the improvements on the notion of causal boundary in [37, 18]. Even though the well-known historical problems of this notion can be minimized in a first approach (as in [30]), finally a consistent notion of the identifications of future and past sets, as well as a reasonable topology, must be carried out. In fact, the former may lead to new interpretations (in order to go *beyond infinity*, as claimed in [36]) and the latter is unavoidable to speak on the *dimension* of the boundary. What is more, the new Busemann-type functions b^\pm here introduced seem to have general applicability for this notion of causal boundary.

Summing up, this work has obvious contents for classical Causality and General Relativity, and it is also introduced as a tool for the string community, in order to check the exact possibilities of holography on plane waves backgrounds.

A. Technical bounds for functionals

Theorem 7. *Let M be a Riemannian manifold and $x_m : [0, \Delta_m] \rightarrow M$ a sequence of piecewise smooth curves with diverging energies and such that the endpoints $x_m(0), x_m(\Delta_m)$ are contained in a bounded region B of M for all m . Then, for any $\lambda < \pi^2$, and any $\mu, k \in \mathbb{R}, 0 < \epsilon < 2$:*

$$\Delta_m \int_0^{\Delta_m} |\dot{x}_m(s)|^2 ds - \frac{1}{\Delta_m} \int_0^{\Delta_m} (\lambda |x_m(s)|^2 + \mu |x_m(s)|^{2-\epsilon} + k) ds \rightarrow \infty.$$

¹²For this subcase and the cases below, assume M complete.

¹³It is sufficient for this asymptotic behaviour to hold in a spatial direction of M if $|F|$ is at most quadratic. For other generalizations, see formula (5.9) and remark 5.

Qualitative F	Causality	Boundary ∂M	Some examples
F superquad. $-F$ at most quad.	No distinguishing	No boundary	pp-waves yielding Sine-Gordon string and related ones
At most quad. F (resp. ¹² $ F $)	Strongly causal	Computable from theorem 2 (resp. theorem 3)	all below
λ -asyp. quad. ¹³ $\lambda > 1/2$	Strongly causal	1-dimension, lightlike	plane waves with some eigenv. $\mu_1 \geq \lambda^2/(1+u^2)$ for $ u $ large
λ -asyp. quad. $\lambda \leq 1/2$	Strongly causal	Critical	pp-wave with $F(x, u) = \lambda^2 x^2/(1+u)^2$ (for $u > 0$)
Subquadratic	Globally hyperbolic	No identif. in $\hat{\partial M}, \check{\partial M}$ Expected higher dim.	(1) \mathbb{L}^n and static type Mp-waves (2) plane waves with $-F$ quadratic

Table 1: Rough properties of the causal boundary of a Mp-wave depending on the qualitative behaviour of F .

Moreover, if the assumption on the endpoints is done only for the initial ones (i.e., $\{x_m(\Delta_m)\}_m$ does not lie necessarily in a bounded B) then the same assertion holds for $\lambda < \pi^2/4$.

Proof. For each m , take the variable $\bar{s} = s/\Delta_m$, $\bar{x}_m(\bar{s}) = x_m(\Delta_m \bar{s})$ and write the corresponding expression (up to a factor 2) as a typical Lagrangian type kinetic minus potential energy:

$$\frac{1}{2} \int_0^1 |\dot{\bar{x}}_m(\bar{s})|^2 d\bar{s} - \int_0^1 \left(\frac{\lambda}{2} |\bar{x}_m(\bar{s})|^2 + (\text{lower degree terms}) \right) d\bar{s}.$$

If the endpoints of the curves were two fixed points, then lemma 3.4 and remark 3.3 in [12] would yield the first assertion. Otherwise, the result follows by connecting all the endpoints to a fixed point by means of curves with bounded energy, and applying previous case.

For the last assertion, just apply the first one to the sequence of curves:

$$\hat{x}_m(s) = \begin{cases} x_m(2s) & \text{if } 0 < s < \Delta_m/2 \\ x_m(2\Delta_m - 2s) & \text{if } \Delta_m/2 < s < \Delta_m. \end{cases}$$

■

Notice that the value of μ in previous result becomes irrelevant (as $\epsilon > 0$), but the inequality for the leading coefficient $\lambda < \pi^2$ or $\lambda < \pi^2/4$ (the optimal ones coming from Wirtinger's Inequality) must hold. Nevertheless, such a bound for λ can be avoided in

the following cases. In particular, the results are stated with $\mu = k = 0$ without loss of generality.

Corollary 2. *Let M be a Riemannian manifold and $x : [0, \Delta_\infty) \rightarrow M$ a piecewise smooth curve with $\Delta_\infty < \infty$ and infinite energy. Then, for A_Δ as in (7.13):*

$$\lim_{\Delta \nearrow \Delta_\infty} A_\Delta = \int_0^{\Delta_\infty} |\dot{x}(s)|^2 ds - R \int_0^{\Delta_\infty} |x(s)|^2 ds = \infty.$$

Even more, for any $K > 0$

$$\lim_{\Delta \nearrow \Delta_\infty} (A_\Delta - K|x_\Delta|^2) = \infty.$$

Proof. For $\delta \in (0, \Delta_\infty)$, put

$$x_\delta(\bar{s}) = x(\delta + (\Delta - \delta)\bar{s}) \quad \forall \bar{s} \in [0, 1]$$

and

$$\begin{aligned} A_\Delta &= A_\delta + \int_\delta^\Delta |\dot{x}(s)|^2 ds - R \int_\delta^\Delta |x(s)|^2 ds \\ &= A_\delta + \frac{1}{\Delta - \delta} \int_0^1 |\dot{x}_\delta(\bar{s})|^2 d\bar{s} - (\Delta - \delta)R \int_0^1 |x_\delta(\bar{s})|^2 d\bar{s} \\ &\geq A_\delta + \frac{1}{\Delta_\infty - \delta} \int_0^1 |\dot{x}_\delta(\bar{s})|^2 d\bar{s} - (\Delta_\infty - \delta)R \int_0^1 |x_\delta(\bar{s})|^2 d\bar{s}. \end{aligned}$$

Thus, the first assertion follows by taking δ close enough to Δ_∞ in order to apply theorem 7 (with $\Delta_m \equiv 1$), i.e., $\Delta_\infty - \delta < \text{Min}\{1, \pi^2/4R\}$.

For the last part, exploiting that $R, K > 0$ are arbitrary, it is enough to check that $\int_0^\Delta |\dot{x}(s)|^2 ds - K|x_\Delta|^2$ is lower bounded for any $K > 0$. Notice that, for $0 < \Delta_0 < \Delta$:

$$(|x_\Delta| - |x_{\Delta_0}|)^2 \leq \left(\int_{\Delta_0}^\Delta |\dot{x}(s)| ds \right)^2 \leq (\Delta - \Delta_0) \int_{\Delta_0}^\Delta |\dot{x}(s)|^2 ds \leq (\Delta_\infty - \Delta_0) \int_{\Delta_0}^\Delta |\dot{x}(s)|^2 ds.$$

Thus, the result follows easily by taking Δ_0 so that $\Delta_\infty - \Delta_0 < 1/2K$. ■

Proposition 8. *Let M be a Riemannian manifold and $R_1 \geq 0, R_2 \in \mathbb{R}, 0 < \epsilon < 2$. There exists $\delta > 0$, which can be taken $\delta = \infty$ if $R_1 = 0$, such that*

$$\int_0^{\bar{\Delta}} |\dot{y}(s)|^2 ds - \int_0^{\bar{\Delta}} (R_1|y(s)|^2 + R_2|y(s)|^{2-\epsilon}) ds > 0$$

for all $\bar{\Delta} \in (0, \delta)$, $y \in \mathcal{C}(x_0, \bar{x}_0; \bar{\Delta})$ and $x_0, \bar{x}_0 \in M$.

Proof. For simplicity, the proof will be carried out with $R_2 = 0$, being obvious the extension to the case $R_2 \neq 0$. First, putting $\tilde{y}(\bar{s}) = y(\bar{\Delta}\bar{s})$:

$$\bar{\Delta} \int_0^{\bar{\Delta}} |\dot{y}(s)|^2 ds - \frac{\pi^2}{2\bar{\Delta}} \int_0^{\bar{\Delta}} |y(s)|^2 ds = \int_0^1 |\dot{\tilde{y}}(\bar{s})|^2 d\bar{s} - \frac{\pi^2}{2} \int_0^1 |\tilde{y}(\bar{s})|^2 d\bar{s} \geq 0,$$

the latter by Wirtinger's inequality. Thus,

$$R_1 \int_0^{\bar{\Delta}} |y(s)|^2 ds \leq \frac{2R_1\bar{\Delta}^2}{\pi^2} \int_0^{\bar{\Delta}} |\dot{y}(s)|^2 ds,$$

and the required inequality follows obviously if $\delta \leq \pi/\sqrt{2R_1}$. ■

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